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New Tensor Spherical Harmonics, for Application to the Partial Differential Equations of Mathematical Physics

R. W. James

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NEW TENSOR SPHERICAL HARMONICS, FOR APPLICATION TO THE PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS

By R. W. JAMES

Department of Applied Mathematics, University of Sydney, Australia 2006

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CONTENTS

Rank-k cartesian-tensor spherical harmonics are defined recursively by the Clebsch–Gordan coupling of rank-$(k-1)$ tensor spherical harmonics with certain complex basis vectors. By taking the rank-0 tensor harmonics to be the usual scalar spherical harmonics, the new definition generates rank-1 harmonics equivalent to the vector spherical harmonics commonly employed in the quantum theory of angular momentum. A second application of the definition generates new rank-2 harmonics which are orthogonal transformations of the symmetric and antisymmetric rank-2 harmonics defined by Zerilli (1970). Continued application of the definition generates new rank-k harmonics which are orthogonally related to tensors used by Burridge (1966). The main advantage of the new tensor harmonics is that the numerous standard properties (for example, completeness; orthogonality; gradient, divergence and curl formulae; addition formulae) of scalar and vector spherical harmonics, generalize, essentially unchanged in form, to the rank-k case. Furthermore, the recursive definition allows systematic evaluation of integrals of products of three tensor harmonics in terms of Wigner coefficients, the latter immediately implying selection rules and symmetries for the integrals. Together, these generalized properties and coupling integrals permit straightforward spherical harmonic analysis of many partial differential equations in mathematical physics. Application of the new harmonics is demonstrated by analysis of the tensor equations of Laplace and Helmhotz, stress–strain equations for free vibrations of an elastic sphere, the Euler and Navier–Stokes equations for a rotating fluid, and the magnetic induction and mean-field magnetic-induction equations for a conducting fluid. Finally, the method of Orszag (1970) for the fast computation of spherical harmonic coefficients of nonlinear interactions is generalized for the tensor-harmonic case.

1. Introduction

The use of spherical harmonic expansions to solve partial differential equations in spherical coordinates, is common to a wide variety of fields. The partial differential equations are thereby reduced to spectral form – a spherical analogue of the Fourier transform method. Common examples are the spectral forms of the equations of fluid mechanics, particularly in meteorological and geomagnetic-dynamo contexts (Elsasser 1946; Takeuchi & Shimazu 1953; Bullard & Gellman 1954; Silberman 1954; Kubota 1960; Baer & Platzman 1961; Merilees 1968; Elsaesser 1966*a*, *b*, 1968; Gibson, Roberts & Scott 1969; Lilley 1970; Kropachev 1971; Roberts 1972; Gubbins 1973; Yabushita 1973; Pekeris, Accad & Schkoller 1973; Frazer 1974). In the past, the task of constructing spectral equations has often been non-systematic, time-consuming and algebraically cumbersome if not intractable. The present paper describes a systematic and concise method for constructing spectral equations. For example, the spectral form of the Navier–Stokes equation is difficult to construct by other methods, and only special cases appear in the literature (see §7 (*d*)). However, the spectral form of the general Navier–Stokes equation becomes almost self-evident once the background theory of the present paper is mastered. This theory is based on a new definition for arbitrary-rank cartesian-tensor spherical harmonics. The definition has the advantage that numerous known properties of scalar and vector spherical harmonics generalize, essentially unchanged in form, to the arbitrary-rank case. The method relies heavily on the properties of Wigner coefficients, and to understand the method in detail, the reader will require some knowledge of 3-*j*, 6-*j* and 9-*j* coefficients. Concise but comprehensive lists of properties can be found in the texts of Brink & Satchler (1968), and Rotenberg, Bivins, Metropolis & Wooten (1959).

The use of Wigner coefficients to form spectral equations is not unique to this paper, having been previously applied to the vector equations of atmospheric oscillations (Jones 1970) and magnetic induction (James 1974). But the method of the present paper allows straightforward inclusion of physical tensors such as stress, anisotropic electrical conductivity, anisotropic mean-field-electrodynamics α-effect, etc. Furthermore, the method allows a simpler approach than otherwise possible to commonly occurring vector and scalar fields which are contractions of higher order tensors – the inertial and viscous terms in the Navier–Stokes equations being examples. Section 7 of this paper illustrates the method applied to the tensor Laplace and Helmholtz equations, and equations of elasticity, fluid mechanics and mean-field magneto-fluidmechanics.

Before proceeding, note that the use of spectral equations is not always a practical way of solving partial differential equations. The spectral method is ideal for a solution which is only a small perturbation about a state of spherical symmetry. But on other occasions, a large number of spherical harmonics may be required to adequately represent a solution, or to determine whether or not a solution exists. Such convergence problems have become notorious in geomagnetic dynamo theory (Gibson *et al.* 1969; Lilley 1970; Gubbins 1973), and only recently have Roberts (1972), Gubbins (1973) and Pekeris *et al.* (1973) shown the spherical harmonic approach to be viable in some circumstances. The obvious alternative to spherical harmonics as a numerical method is a finite-difference grid covering all independent variables. The spectral method has several important advantages. No mapping of the sphere is needed, so that difficulties of finite-difference methods near the poles are eliminated, angular derivatives are treated exactly, and some boundary and derivative conditions can be automatically satisfied. In addition, conservation laws are preserved and aliasing errors blocked. These advantages are discussed in detail by Platzman (1960), Baer & Platzman (1961), Elsaesser (1966*a*) and Merilees (1968). A direct comparison by Elsaesser (1966*a*) found the spectral method to be about twice as efficient in obtaining a given level of accuracy when integrating certain weather prediction equations. This result is not generally true. Indeed, until recently, a major drawback to the spectral method was the amount of time required to compute the spectral coefficients of nonlinear interactions. However, Orszag (1970) has shown how to significantly reduce this amount of computation to be comparable with finite-difference methods. Section 8 of the present paper shows how to generalize Orszag's results to representations which use the new tensor spherical harmonics.

2. Definition of the tensor harmonics

Consider the complex reference vectors $\boldsymbol{e}_0 = \boldsymbol{e}_z$, $\boldsymbol{e}_1 = -2^{-\frac{1}{2}}(\boldsymbol{e}_x + \mathrm{i}\boldsymbol{e}_y)$, $\boldsymbol{e}_{-1} = 2^{-\frac{1}{2}}(\boldsymbol{e}_x - \mathrm{i}\boldsymbol{e}_y)$, where $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_z$ are cartesian reference vectors. The general rank-k tensor surface harmonic $\boldsymbol{Y}^{m_0}_{n_0,\ldots,n_k}$, a function of colatitude θ and east-longitude ϕ, will be defined recursively by the Clebsch–Gordan coupling of related rank-$(k-1)$ tensor harmonics and $\boldsymbol{e}_\mu$ $(\mu = 0, \pm 1)$. Specifically,

$$\boldsymbol{Y}^{m_0}_{n_0,\ldots,n_k} = (-)^{n_0-m_0}(2n_0+1)^{\frac{1}{2}} \sum_{m_1,\,\mu} \begin{pmatrix} n_0 & n_1 & 1 \\ m_0 & -m_1 & -\mu \end{pmatrix} \boldsymbol{Y}^{m_1}_{n_1,\ldots,n_k} \boldsymbol{e}_\mu, \tag{2.1}$$

where the rank-$(k-1)$ tensor harmonic $\boldsymbol{Y}^{m_1}_{n_1,\ldots,n_k}$ is similarly defined in terms of rank-$(k-2)$ tensor harmonics $\boldsymbol{Y}^{m_2}_{n_2,\ldots,n_k}$.

The 2×3 array in (2.1) is a Wigner 3-j coefficient, and is zero unless $|m_0| \leqslant n_0$, $|m_1| \leqslant n_1$, $|\mu| \leqslant 1$, and $m_1 = m_0 - \mu$. Thus the summation in (2.1) contains only three terms, and, for given $n_0, \ldots, n_k$ there are $2n_0+1$ tensor harmonics corresponding to $m_0 = -n_0, \ldots, n_0$. The 3-j coefficient in (2.1) is also zero unless n_0, n_1, 1 satisfy the triangle inequality $|n_0-1| \leqslant n_1 \leqslant |n_0+1|$. The recursive nature of (2.1) thus implies that, for a non-trivial harmonic,

$$\begin{gathered} |n_1 - 1| \leqslant n_2 \leqslant n_1 + 1, \\ |n_2 - 1| \leqslant n_3 \leqslant n_2 + 1, \\ \vdots \\ |n_{k-1} - 1| \leqslant n_k \leqslant n_{k-1} + 1. \end{gathered}$$

So, for given n_0 and m_0, there are at most 3^k distinct non-zero rank-k tensor harmonics defined by (2.1).

For conciseness, it will be useful to introduce some abbreviations. First, the notation in (2.1) is such that superscript m_i generally corresponds to a first-subscript n_i. Similarly, in later sections, superscripts M_i and $m_{i\alpha}$ correspond to subscripts N_i and $n_{i\alpha}$. Thus, unless such correspondences break down, m-superscripts will be omitted. Secondly, the string of parameters $n_i, n_{i+1}, \ldots, n_{i+k}$ will be abbreviated to $n_i(k)$. Thirdly, zero subscripts will often be omitted so that n and m are to be regarded as identical to n_0 and m_0. Finally, define

$$\Lambda(a, b, c, \ldots) = [(2a+1)\,(2b+1)\,(2c+1)\ldots]^{\frac{1}{2}}.$$

With these abbreviations in mind,

$$(-)^{m+m_0} \equiv 1,$$

$$\Lambda(n(k)) \equiv [(2n+1)\,(2n_1+1)\,\dots\,(2n_k+1)]^{\frac{1}{2}},$$

and definition (2.1) may be rewritten as

$$\boldsymbol{Y}_{n(k)} = (-)^{n-m}\Lambda(n) \sum_{m_1,\,\mu} \begin{pmatrix} n & n_1 & 1 \\ m & -m_1 & -\mu \end{pmatrix} \boldsymbol{Y}_{n_1(k-1)}\,\boldsymbol{e}_\mu. \tag{2.2}$$

As a starting point for the recursive definition, the rank-0 harmonics $Y_{n(0)}$ will be identified with the usual scalar complex spherical surface harmonics $Y_n^m(\theta, \phi)$, normalized as in James (1973). Equation (2.2) then defines vector spherical harmonics $\boldsymbol{Y}_{n(1)}$ identical to the $\boldsymbol{Y}^m_{n,\,n_1}$ of James (1973) and the $\boldsymbol{Y}_{n,\,n_1,\,m}$ of Edmonds (1957), and $(4\pi)^{\frac{1}{2}}$ times the $\boldsymbol{Y}^m_{n,\,n_1,\,1}$ of Brink & Satchler (1968). Continued application of (2.2) generates higher order tensor harmonics in terms of the $\boldsymbol{e}_\mu$ basis vectors. The general result, easily derived by induction over k, is

$$\boldsymbol{Y}_{n(k)} = \sum_{\substack{m_1,\,\dots,\,m_k \\ \mu_1,\,\dots,\,\mu_k}} K Y^{m_k}_{n_k}\,\boldsymbol{e}_{\mu_k} \dots \boldsymbol{e}_{\mu_1}, \tag{2.3}$$

where
$$K = \prod_{i=1}^{k} (-)^{n_{i-1}-m_{i-1}} \Lambda(n_{i-1}) \begin{pmatrix} n_{i-1} & n_i & 1 \\ m_{i-1} & -m_i & -\mu_i \end{pmatrix}. \tag{2.4}$$

The result (2.3) is useful for evaluating the cartesian components of $\boldsymbol{Y}_{n(k)}$, and shows that $\boldsymbol{Y}_{n(k)}$ depends on scalar harmonics of degree n_k only.

In special cases, the constant K in (2.3) is easily evaluated. In particular, we find

$$\boldsymbol{Y}^\mu_{n,\,0} = \delta^1_n\,\boldsymbol{e}_\mu, \tag{2.5}$$

$$\boldsymbol{Y}^0_{0,\,1,\,0} = -\boldsymbol{\delta}/\surd 3,$$

$$\boldsymbol{Y}^0_{0,\,1,\,1,\,0} = -\mathrm{i}\boldsymbol{\varepsilon}/\surd 6,$$

where δ^1_n is the Kronecker delta;

$$\boldsymbol{\delta} = \sum_\mu (-)^\mu\,\boldsymbol{e}_\mu\,\boldsymbol{e}_{-\mu} \tag{2.6}$$

is the unit dyadic; and

$$\boldsymbol{\varepsilon} = \mathrm{i}\surd 6 \sum_{\mu_1,\,\mu_2,\,\mu_3} \begin{pmatrix} 1 & 1 & 1 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix} \boldsymbol{e}_{\mu_3}\,\boldsymbol{e}_{\mu_2}\,\boldsymbol{e}_{\mu_1}$$

is the rank-3 alternating tensor.

3. Basic properties

Starting with the known properties of the scalar and vector harmonics $Y_{n(0)}$ and $\boldsymbol{Y}_{n(1)}$, the recursive definition (2.2) implies, by induction over k, a multitude of useful properties for the rank-k tensor harmonic $\boldsymbol{Y}_{n(k)}$. The sample list of properties given in this section was derived from known properties in the works of Edmonds (1957), Brink & Satchler (1968), and James (1974). These known properties can be retrieved by putting $k = 0$ and identifying n_0 and m_0 with n and m. Only brief outlines of proofs will be given, detailed proofs being worthwhile exercises for interested readers. Throughout the remainder of this article, the symbols †, ¯ and * represent transpose, complex-conjugate and complex-conjugate-transpose respectively; the generalized dot product is defined by

$$\boldsymbol{e}_{\mu_1} \dots \boldsymbol{e}_{\mu_k} \cdot \bar{\boldsymbol{e}}_{\lambda_k} \dots \bar{\boldsymbol{e}}_{\lambda_1} = \delta^{\mu_1}_{\lambda_1} \dots \delta^{\mu_k}_{\lambda_k}; \tag{3.1}$$

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$\mathrm{d}\Omega = \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$; f represents a function of the radial distance r; and

$$\partial_n^{n'} \equiv \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}r} + \dfrac{n+1}{r}, & \text{if} \quad n = n'+1, \\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}r} - \dfrac{n}{r}, & \text{if} \quad n = n'-1, \end{cases}$$

$$D_n \equiv \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{n(n+1)}{r^2},$$

are differential operators acting on f. A rule of thumb worth noting for the differentiation properties which follow, is that the subscript n on ∂ is the last subscript of the tensor harmonic being differentiated, and the superscript n' on ∂ is the last subscript of the tensor harmonic resulting from differentiation.

Orthogonality

The known orthogonality properties of 3-j coefficients and scalar spherical harmonics combine to give the rank-k orthogonality property

$$\frac{1}{4\pi}\int \boldsymbol{Y}_{n(k)}{}^{*}\cdot \boldsymbol{Y}_{N(k)}\,\mathrm{d}\Omega = \delta_{n_0}^{N_0}\ldots\delta_{n_k}^{N_k}\,\delta_m^M. \tag{3.2}$$

Completeness

The orthogonality of 3-j coefficients and the completeness of scalar spherical harmonics allow any rank-k tensor $\boldsymbol{F}_{(k)}(\theta,\phi)$, which is continuous over the unit sphere $(r = 1)$, to be expanded as

$$\boldsymbol{F}_{(k)}(\theta,\phi) = \sum_{n(k),\,m} F_{n(k)}\,\boldsymbol{Y}_{n(k)}(\theta,\phi), \tag{3.3}$$

where, by (3.2),

$$F_{n(k)} = \frac{1}{4\pi}\int \boldsymbol{Y}_{n(k)}{}^{*}\cdot \boldsymbol{F}_{(k)}\,\mathrm{d}\Omega.$$

The summation in (3.3) is for $n = 0, 1, 2, \ldots$; $|m| \leqslant n$; and n_i $(i = 1, \ldots, k)$ satisfying the triangle conditions in § 2. This completeness property may be extended to discontinuous tensors satisfying appropriate Dirichlet conditions (MacMillan 1958, p. 386).

Complex-conjugate

Using an appropriate symmetry property of 3-j coefficients, the known rank-0 result generalizes to

$$\overline{\boldsymbol{Y}_{n(k)}^m} = (-)^{n+n_k+m+k}\,\boldsymbol{Y}_{n(k)}^{-m}. \tag{3.4}$$

An immediate implication of (3.4) is that, if $\boldsymbol{F}_{(k)}$ in (3.3) is a real function,

$$\overline{F_{n(k)}^m} = (-)^{n+n_k+m+k}\,F_{n(k)}^{-m}.$$

Gradient formulae

The standard gradient formula generalizes by straightforward induction to

$$\nabla[f\boldsymbol{Y}_{n(k)}] = \sum_{n_{k+1}} G(n_k, n_{k+1})\,\boldsymbol{Y}_{n(k+1)}\,\partial_{n_k}^{n_{k+1}} f, \tag{3.5}$$

where
$$G(n_k, n_{k+1}) = (-)^{n_k}\Lambda(n_{k+1})\begin{pmatrix} n_k & n_{k+1} & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Selection rules for 3-j coefficients restrict the sum in (3.5) to at most two terms corresponding to $n_{k+1} = n_k \pm 1$. Evaluation of the relevant 3-j coefficients gives

$$G(n_k, n_{k+1}) = \frac{1}{(2n_k+1)^{\frac{1}{2}}} \begin{cases} n_k^{\frac{1}{2}}, & \text{if} \quad n_{k+1} = n_k - 1, \\ -(n_k+1)^{\frac{1}{2}}, & \text{if} \quad n_{k+1} = n_k + 1, \\ 0, & \text{otherwise.} \end{cases} \tag{3.6}$$

The special rank-0 case with $f = r^n$ generalizes to

$$\nabla[r^{n_k} \boldsymbol{Y}_{n(k)}] = [n_k(2n_k+1)]^{\frac{1}{2}} r^{n_k-1} \boldsymbol{Y}_{n(k),\, n_k-1}; \tag{3.7}$$

which, in turn, generalizes to

$$\overbrace{\nabla \ldots \nabla}^{l \text{ times}} [r^{n_k} \boldsymbol{Y}_{n(k)}] = \left[\frac{(2n_k+1)\,!}{2^l(2n_k-2l+1)\,!}\right]^{\frac{1}{2}} r^{n_k-l} \boldsymbol{Y}_{n(k),\, n_k-1, \ldots,\, n_k-l}.$$

The special rank-0 case with $f = r^{-n-1}$ generalizes to

$$\nabla[r^{-n_k-1} \boldsymbol{Y}_{n(k)}] = [(n_k+1)\,(2n_k+1)]^{\frac{1}{2}} r^{-n_k-2} \boldsymbol{Y}_{n(k),\, n_k+1}, \tag{3.8}$$

which, in turn, generalizes to

$$\overbrace{\nabla \ldots \nabla}^{l \text{ times}} [r^{-n_k-1} \boldsymbol{Y}_{n(k)}] = \left[\frac{(2n_k+2l)\,!}{2^l(2n_k)\,!}\right]^{\frac{1}{2}} r^{-n_k-l-1} \boldsymbol{Y}_{n(k),\, n_k+1, \ldots,\, n_k+l}.$$

L-operator

Let $\boldsymbol{L} \equiv -\mathrm{i}\boldsymbol{r} \times \nabla$, where $\boldsymbol{r}$ is the radial position vector. Then, by straightforward induction from the known rank-0 result,

$$\boldsymbol{L}\boldsymbol{Y}_{n(k)} = [n_k(n_k+1)]^{\frac{1}{2}} \boldsymbol{Y}_{n(k),\, n_k}. \tag{3.9}$$

Divergence

The standard divergence formula generalizes by straightforward induction to

$$\nabla \cdot [f\boldsymbol{Y}_{n(k+1)}] = G(n_k, n_{k+1})\, \boldsymbol{Y}_{n(k)}\, \partial^{n_k}_{n_{k+1}} f, \tag{3.10}$$

where G is given by (3.6). We note, in particular, that the rank-k tensor-harmonics with $n_{k+1} = n_k$ – there are at most 3^{k-1} of them – are always solenoidal.

Curl

The general rank-$(k+1)$ result is

$$\nabla \times [f\boldsymbol{Y}_{n(k+1)}] = \sum_{n'_{k+1}} C(n_k, n_{k+1}, n'_{k+1})\, \boldsymbol{Y}_{n(k),\, n'_{k+1}}\, \partial^{n'_{k+1}}_{n_{k+1}} f, \tag{3.11}$$

where $$C(n_k, n_{k+1}, n'_{k+1}) = (-)^{n_k+n'_{k+1}} \mathrm{i}\sqrt{6}\, \Lambda(n_{k+1})\, G(n_{k+1}, n'_{k+1}) \begin{Bmatrix} n_k & n_{k+1} & 1 \\ 1 & 1 & n'_{k+1} \end{Bmatrix}. \tag{3.12}$$

The 2×3 array in (3.12) is a Wigner 6-j coefficient. Triangle inequalities for 3-j and 6-j coefficients restrict the sum in (3.11) to at most 2 terms, corresponding to $n'_{k+1} = n_{k+1} \pm 1$. Explicit evaluation of the relevant 3-j and 6-j coefficients gives

$$\left.\begin{aligned} \nabla \times [f\boldsymbol{Y}_{n(k),\, n_k-1}] &= \mathrm{i}\left(\frac{n_k+1}{2n_k+1}\right)^{\frac{1}{2}} \boldsymbol{Y}_{n(k),\, n_k}\, \partial^{n_k}_{n_k-1} f, \\ \nabla \times [f\boldsymbol{Y}_{n(k),\, n_k}] &= \frac{\mathrm{i}}{(2n_k+1)^{\frac{1}{2}}} [n_k^{\frac{1}{2}} \boldsymbol{Y}_{n(k),\, n_k+1}\, \partial^{n_k+1}_{n_k} + (n_k+1)^{\frac{1}{2}} \boldsymbol{Y}_{n(k),\, n_k-1}\, \partial^{n_k-1}_{n_k}]\, f, \\ \nabla \times [f\boldsymbol{Y}_{n(k),\, n_k+1}] &= \mathrm{i}\left(\frac{n_k}{2n_k+1}\right)^{\frac{1}{2}} \boldsymbol{Y}_{n(k),\, n_k}\, \partial^{n_k}_{n_k+1} f. \end{aligned}\right\} \tag{3.13}$$

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The formula (3.13) can be easily derived by straightforward induction from the known rank-0 result. However, formula (3.11) is new, apparently even for the $k = 0$ case, in that the coefficients in (3.13) are recognized to be 6-j coefficients. The proof of (3.11), without merely verifying that it is equivalent to (3.13), is not simple, but can be carried out by combining the Gradient Formula (3.5) with the vector-product coupling integral (4.3) in § 4.

The symmetries of Wigner coefficients allow (3.11) to be written in the alternative form

$$\nabla \times [f\boldsymbol{Y}_{n(k),\, n'_{k+1}}] = \sum_{n_{k+1}} C(n_k, n_{k+1}, n'_{k+1})\, \boldsymbol{Y}_{n(k+1)} \partial^{n_{k+1}}_{n'_{k+1}} f, \tag{3.14}$$

which will be useful for the applications in § 7.

Laplacian

By straightforward induction from the known rank-0 formula,

$$\nabla^2[f\boldsymbol{Y}_{n(k)}] = \boldsymbol{Y}_{n(k)} D_{n_k} f, \tag{3.15}$$

a result also easily derived by combining (3.5) and (3.10) and noting

$$D_{n_k} \equiv \partial^{n_k}_{n_{k+1}} \partial^{n_{k+1}}_{n_k}.$$

Grad–div

Combining (3.5) and (3.10)

$$\nabla\nabla\cdot[f\boldsymbol{Y}_{n(k+1)}] = \sum_{n'_{k+1}} G(n_k, n_{k+1})\, G(n_k, n'_{k+1})\, \boldsymbol{Y}_{n(k),\, n'_{k+1}} \partial^{n'_{k+1}}_{n_k} \partial^{n_k}_{n_{k+1}} f. \tag{3.16}$$

The 3-j triangle inequalities restrict this sum to at most two terms with simple coefficients given by (3.6).

Addition formulae

Let ω be the angle between the directions defined by angle-pairs (θ_1, ϕ_1), (θ_2, ϕ_2). Use of 3-j orthogonality allows generalization of the standard addition formula to

$$\sum_m \boldsymbol{Y}_{n(k)}(\theta_1, \phi_1)^* \cdot \boldsymbol{Y}_{n(k)}(\theta_2, \phi_2) = (2n_k+1)\, P_{n_k}(\cos\omega),$$

and

$$\sum_{\substack{n(k-1)\\ m}} \boldsymbol{Y}_{n(k)}(\theta_1, \phi_1)^*\, \boldsymbol{Y}_{n(k)}(\theta_2, \phi_2) = (2n_k+1)\, P_{n_k}(\cos\omega)\, \boldsymbol{\delta}_{(2k)}.$$

Here,

$$P_n(\cos\omega) = Y^0_n(\omega)/\Lambda(n)$$

is the Legendre polynomial, and

$$\boldsymbol{\delta}_{(2k)} = \sum_{\mu_1, \ldots, \mu_k} (\boldsymbol{e}_{\mu_1} \ldots \boldsymbol{e}_{\mu_k})^* (\boldsymbol{e}_{\mu_1} \ldots \boldsymbol{e}_{\mu_k})$$

is a rank-$2k$ generalization of the unit dyadic in (2.6).

Rotation of coordinate frame

Introduce Euler angles α, β, γ and rotation matrix elements

$$D^n_{M,m}(-\gamma, -\beta, -\alpha) = \overline{D^n_{m,M}}(\alpha, \beta, \gamma),$$
$$= d^n_{m,M}(\beta)\, \mathrm{e}^{\mathrm{i}(m\alpha + M\gamma)},$$

defined as in Brink & Satchler (1968). Consider a new reference frame, defined by complex reference vectors $\boldsymbol{e}'_\mu$ $(\mu = 0, \pm 1)$, obtained by rotating the reference vectors $\boldsymbol{e}_\mu$ $(\mu = 0, \pm 1)$

through the angles α, β, γ (Brink & Satchler 1968, Figure 2). Let the point (r, θ, ϕ) have coordinates (r, θ', ϕ') in the new frame, and let $\boldsymbol{Y}'_{n(k)}$ be a rank-k tensor harmonic at (θ', ϕ') in the new frame. That is

$$\boldsymbol{Y}'_{n(k)} = (-)^{n-m} \Lambda(n) \sum_{\mu, m_1} \begin{pmatrix} n & n_1 & 1 \\ m & -m_1 & -\mu \end{pmatrix} \boldsymbol{Y}'_{n_1(k-1)} \boldsymbol{e}'_{\mu},$$

where $Y'_{n(0)} = Y_n^m(\theta', \phi')$.

The standard contraction properties of rotation matrix elements allow the known relation between $Y'_{n(0)}$ and $Y_{n(0)}$ to be generalized to the rank-k case:

$$\boldsymbol{Y}^m_{n(k)} = \sum_M D^n_{M,m}(-\gamma, -\beta, -\alpha)\, \boldsymbol{Y}^{M\prime}_{n(k)}, \tag{3.17}$$

$$\boldsymbol{Y}^{M\prime}_{n(k)} = \sum_m D^n_{m,M}(\alpha, \beta, \gamma)\, \boldsymbol{Y}^m_{n(k)}. \tag{3.18}$$

In rotation-group jargon, (3.17) and (3.18) mean that the $2n+1$ quantities $\boldsymbol{Y}^m_{n(k)}$ $(m = -n, \ldots, n)$ constitute an irreducible *spherical* tensor of rank-n; that is, a basis for an irreducible representation of the rotation group.

Polar components

Suppose that in the preceding rotation, we choose $\alpha = \phi$, $\beta = \theta$, $\gamma = 0$. Then $\boldsymbol{e}'_0 = \boldsymbol{e}_r$ and $\boldsymbol{e}'_1 = -\bar{\boldsymbol{e}}'_{-1} = -2^{-\frac{1}{2}}(\boldsymbol{e}_\theta + \mathrm{i}\boldsymbol{e}_\phi)$, where $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\phi$ are the unit vectors in the directions of increasing r, θ, ϕ. Also, $\theta' = 0$, so that

$$Y_n^m(\theta', \phi') = \Lambda(n)\, \delta^0_m.$$

Combining (3.17) with the rotated version of (2.3) and using standard symmetry properties of rotation matrix elements leads to

$$\boldsymbol{Y}_{n(k)} = \sum_{\mu_1, \ldots, \mu_k} K'\, d^n_{m,M}(\theta)\, \mathrm{e}^{\mathrm{i}m\phi}\, \boldsymbol{e}'_{\mu_k} \ldots \boldsymbol{e}'_{\mu_1}. \tag{3.19}$$

Here
$$K' = \Lambda(n_k) \prod_{i=1}^{k} (-)^{n_{i-1}+M_{i-1}} \Lambda(n_{i-1}) \begin{pmatrix} n_{i-1} & n_i & 1 \\ M_{i-1} & -M_i & -\mu_i \end{pmatrix}, \tag{3.20}$$

with
$$\begin{aligned} M = M_0 &= \mu_1 + \ldots + \mu_k, \\ M_1 &= \mu_2 + \ldots + \mu_k, \\ &\vdots \\ M_{k-1} &= \mu_k, \\ M_k &= 0. \end{aligned}$$

Equation (3.19) shows that each polar component of $\boldsymbol{Y}_{n(k)}$ is proportional to $d^n_{m,M}(\theta)\, \mathrm{e}^{\mathrm{i}m\phi}$, a quantity occasionally referred to as a 'generalized spherical harmonic'. This generalized spherical harmonic, and hence the polar components of $\boldsymbol{Y}_{n(k)}$, can be rewritten in terms of the derivatives of the scalar spherical harmonic Y_n^m. For $k > 2$ it is tedious and of no advantage to do this, but appendix A illustrates the relatively simple and commonly occurring cases $k = 1$ and $k = 2$.

Alternatively, the polar components of $\boldsymbol{Y}_{n(k)}$ can be found by multiple applications of the operators ∇ and $\boldsymbol{L}$, in polar form, using properties (3.7), (3.8), (3.9). This procedure is very tedious for $k \geqslant 2$, but was used to check appendix A.

Equation (3.19) facilitates the evaluation of polar components, and also the application of boundary conditions involving polar components. Two useful byproducts of (3.19) are

$$\boldsymbol{e}_r \boldsymbol{Y}_{n(k)} = \sum_{n_{k+1}} G(n_k, n_{k+1})\, \boldsymbol{Y}_{n(k+1)}, \tag{3.21}$$

$$\boldsymbol{e}_r \cdot \boldsymbol{Y}_{n(k+1)} = G(n_k, n_{k+1})\, \boldsymbol{Y}_{n(k)}. \tag{3.22}$$

The G-coefficients in (3.21) and (3.22) are given simply by (3.6), implying in particular that the sum in (3.21) contains at most the two terms corresponding to $n_{k+1} = n_k \pm 1$. Correspondingly, (3.22) implies that 3^k of the rank-$(k+1)$ tensor harmonics, namely those with $n_{k+1} = n_k$, are tangential (on the left) to spheres centred at $r = 0$. But such harmonics are also solenoidal according to (3.10). They are indeed the rank-$(k+1)$ generalization of the toroidal vector harmonics often used in the representations of divergence-free vector fields (see, for example, Bullard & Gellman 1954).

The foregoing list is not exhaustive. For example, it could be extended to include tensor-harmonic expansions of Dirac delta functions, inverse radii, spherical waves, etc. However, the properties listed illustrate how generalization to the rank-k case is effected, and are sufficient for the applications in the remaining sections of this paper.

4. Coupling integrals

The expansion of products of tensor functions in series of tensor harmonics requires evaluation of integrals of products of three tensor harmonics. We will follow the notation of Bullard & Gellman (1954) and distinguish different components of such products by subscripts α, β, γ. Thus, extending the notation of earlier sections, $n_\alpha(k)$ will denote the string $n_\alpha, n_{1\alpha}, \ldots, n_{k\alpha}$; and $\boldsymbol{Y}_{n_\alpha(k)}$ a rank-k tensor harmonic.

The most basic integral of this type, evaluated independently by Adams (1900) and Gaunt (1929), is, in terms of Wigner coefficients,

$$\frac{1}{4\pi}\int Y_{n_\alpha(0)} Y_{n_\beta(0)} Y_{n_\gamma(0)}\,\mathrm{d}\Omega = \Lambda(n_\alpha, n_\beta, n_\gamma)\begin{pmatrix} n_\alpha & n_\beta & n_\gamma \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} n_\alpha & n_\beta & n_\gamma \\ m_\alpha & m_\beta & m_\gamma \end{pmatrix}. \tag{4.1}$$

The integral (4.1) is useful for expanding a product of two scalar functions. Two important related integrals are

$$\frac{1}{4\pi}\int \boldsymbol{Y}_{n_\alpha(1)}\cdot\boldsymbol{Y}_{n_\beta(1)} Y_{n_\gamma(0)}\,\mathrm{d}\Omega = (-)^{n_\alpha+n_{1\alpha}}\Lambda(n_\alpha(1), n_\beta(1), n_\gamma)$$
$$\times\begin{Bmatrix} n_\alpha & n_{1\alpha} & 1 \\ n_{1\beta} & n_\beta & n_\gamma \end{Bmatrix}\begin{pmatrix} n_{1\alpha} & n_{1\beta} & n_\gamma \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} n_\alpha & n_\beta & n_\gamma \\ m_\alpha & m_\beta & m_\gamma \end{pmatrix}, \tag{4.2}$$

and
$$\frac{1}{4\pi}\int \boldsymbol{Y}_{n_\alpha(1)}\times\boldsymbol{Y}_{n_\beta(1)}\cdot\boldsymbol{Y}_{n_\gamma(1)}\,\mathrm{d}\Omega = (-)^{n_\alpha+n_\beta+n_\gamma+1}6^{\frac{1}{2}}\mathrm{i}\Lambda(n_\alpha(1), n_\beta(1), n_\gamma(1))$$
$$\times\begin{Bmatrix} n_\alpha & n_\beta & n_\gamma \\ n_{1\alpha} & n_{1\beta} & n_{1\gamma} \\ 1 & 1 & 1 \end{Bmatrix}\begin{pmatrix} n_{1\alpha} & n_{1\beta} & n_{1\gamma} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} n_\alpha & n_\beta & n_\gamma \\ m_\alpha & m_\beta & m_\gamma \end{pmatrix}, \tag{4.3}$$

where the 3×3 array in (4.3) is a Wigner 9-j coefficient. The integral (4.2), evaluated by Jones (1970), is useful for expanding dot products of vector functions in series of scalar harmonics, or products of scalar and vector functions in series of vector harmonics. The integral (4.3), evaluated by James (1974), is useful for expanding cross-products of vector functions in series of vector harmonics.

We are more interested here in coupling-integrals involving tensor harmonics of higher rank. General formula in terms of Wigner coefficients are readily obtained by combining (3.19) with the known result

$$\frac{1}{4\pi}\int d^{n_\alpha}_{m_\alpha, M_\alpha}(\theta)\, d^{n_\beta}_{m_\beta, M_\beta}(\theta)\, d^{n_\gamma}_{m_\gamma, M_\gamma}(\theta)\,\mathrm{e}^{\mathrm{i}(m_\alpha+m_\beta+m_\gamma)\phi}\,\mathrm{d}\Omega = \begin{pmatrix} n_\alpha & n_\beta & n_\gamma \\ m_\alpha & m_\beta & m_\gamma \end{pmatrix}\begin{pmatrix} n_\alpha & n_\beta & n_\gamma \\ M_\alpha & M_\beta & M_\gamma \end{pmatrix}, \tag{4.4}$$

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which is valid provided

$$M_\alpha + M_\beta + M_\gamma = 0. \tag{4.5}$$

Constraint (4.5) is always satisfied, merely reflecting that all coupling integrals are scalar quantities. However, simpler formulae can be obtained by using definition (2.2) to generate coupling integrals recursively. This statement is illustrated below for the most common cases; that is, where tensors are contracted to form scalars or vectors.

We will use notation such as

$$(n_\alpha(k);\, m_\alpha \cdot n_\beta(k);\, m_\beta \cdot n_\gamma(0);\, m_\gamma) = \frac{1}{4\pi}\int \mathbf{Y}_{n_\alpha(k)} \cdot \mathbf{Y}_{n_\beta(k)} Y_{n_\gamma(0)}\, \mathrm{d}\Omega, \tag{4.6}$$

$$(n_\alpha(k);\, m_\alpha \cdot n_\beta(k);\, m_\beta{}^\dagger \cdot n_\gamma(0);\, m_\gamma) = \frac{1}{4\pi}\int \mathbf{Y}_{n_\alpha(k)} \cdot \mathbf{Y}_{n_\beta(k)}{}^\dagger Y_{n_\gamma(0)}\, \mathrm{d}\Omega, \tag{4.7}$$

$$(n_\alpha(k);\, m_\alpha \cdot n_\beta(k-1);\, m_\beta \cdot \overline{n_\gamma(1);\, m_\gamma}) = \frac{1}{4\pi}\int \mathbf{Y}_{n_\alpha(k)} \cdot \mathbf{Y}_{n_\beta(k-1)} \cdot \overline{\mathbf{Y}}_{n_\gamma(1)}\, \mathrm{d}\Omega. \tag{4.8}$$

Consistent with the Wigner–Eckart theorem (see, for example, Brink & Satchler 1968), the m-dependence of integrals such as (4.6) and (4.7) is contained in one 3-j coefficient. Furthermore, (2.3) and (4.1) imply that such integrals are proportional to a 3-j coefficient with zero bottom-row. Thus, we can write, for example,

$$(n_\alpha(k);\, m_\alpha \cdot n_\beta(k);\, m_\beta \cdot n_\gamma(0);\, m_\gamma) = (n_\alpha(k)\cdot n_\beta(k)\cdot n_\gamma(0)) \begin{pmatrix} n_{k\alpha} & n_{k\beta} & n_\gamma \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} n_\alpha & n_\beta & n_\gamma \\ m_\alpha & m_\beta & m_\gamma \end{pmatrix}, \tag{4.9}$$

$$(n_\alpha(k);\, m_\alpha \cdot n_\beta(k);\, m_\beta{}^\dagger \cdot n_\gamma(0);\, m_\gamma) = (n_\alpha(k)\cdot n_\beta(k)^\dagger\cdot n_\gamma(0)) \begin{pmatrix} n_{k\alpha} & n_{k\beta} & n_\gamma \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} n_\alpha & n_\beta & n_\gamma \\ m_\alpha & m_\beta & m_\gamma \end{pmatrix},$$

where the coefficients $(n_\alpha(k)\cdot n_\beta(k)\cdot n_\gamma(0))$ and $(n_\alpha(k)\cdot n_\beta(k)^\dagger\cdot n_\gamma(0))$ are independent of m_α, m_β, m_γ. We will call such coefficients 'reduced integrals'. These reduced integrals are analogous to, but an extension of, the concept of 'reduced matrix elements' in the quantum theory of angular momentum. Note that in computations with $(n_\alpha(k)\cdot n_\beta(k)\cdot n_\gamma(0))$, for example, $n_{k\alpha}+n_{k\beta}+n_\gamma$ may be assumed to be even, since otherwise the 3-j coefficient with zero bottom row in (4.9) is zero. Integrals such as (4.8) may be 'reduced' by using (3.4). For example,

$$\begin{aligned}(n_\alpha(k);\, m_\alpha \cdot n_\beta(k-1);\, m_\beta \cdot \overline{n_\gamma(1);\, m_\gamma})\\ &= (-)^{n_\gamma+n_{1\gamma}+m_\gamma+1}(n_\alpha(k);\, m_\alpha \cdot n_\beta(k-1);\, m_\beta \cdot n_\gamma(1);\, -m_\gamma)\\ &= (-)^{n_\gamma+n_{1\gamma}+m_\gamma+1}(n_\alpha(k)\cdot n_\beta(k-1)\cdot n_\gamma(1)) \begin{pmatrix} n_{k\alpha} & n_{(k-1)\beta} & n_{1\gamma} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} n_\alpha & n_\beta & n_\gamma \\ m_\alpha & m_\beta & -m_\gamma \end{pmatrix}.\end{aligned}$$

The simplest recursive relations between the reduced integrals required for contracting tensors to form vectors and scalars are

$$(n_\alpha(k-1)\cdot n_\beta(k)\cdot n_\gamma(1)) = (-)^{n_\alpha+n_\beta+n_{1\gamma}}\Lambda(n_\beta, n_\gamma) \begin{Bmatrix} n_\beta & n_{1\beta} & 1 \\ n_{1\gamma} & n_\gamma & n_\alpha \end{Bmatrix} (n_\alpha(k-1)\cdot n_{1\beta}(k-1)\cdot n_{1\gamma}(0)), \tag{4.10}$$

$$(n_\alpha(k)\cdot n_\beta(k)^\dagger\cdot n_\gamma(0)) = (-)^{n_\alpha+n_{1\beta}+n_\gamma}\Lambda(n_\alpha, n_\beta) \begin{Bmatrix} n_\alpha & n_{1\alpha} & 1 \\ n_{1\beta} & n_\beta & n_\gamma \end{Bmatrix} (n_{1\alpha}(k-1)\cdot n_{1\beta}(k-1)^\dagger\cdot n_\gamma(0)), \tag{4.11}$$

$$(n_\alpha(k)\cdot n_\beta(k-1)^\dagger\cdot n_\gamma(1)) = (-)^{n_\alpha+n_{1\beta}+n_\gamma}\Lambda(n_\alpha, n_\beta) \begin{Bmatrix} n_\alpha & n_{1\alpha} & 1 \\ n_{1\beta} & n_\beta & n_\gamma \end{Bmatrix} (n_{1\alpha}(k-1)\cdot n_{1\beta}(k-2)^\dagger\cdot n_\gamma(1)), \tag{4.12}$$

$$(n_\alpha(k)^\dagger \cdot n_\beta(k-1)\cdot n_\gamma(1)) = (-)^{n_{1\alpha}+n_\beta+n_\gamma}\Lambda(n_\alpha, n_\gamma)\begin{Bmatrix} n_\alpha & n_{1\alpha} & 1 \\ n_{1\gamma} & n_\gamma & n_\beta \end{Bmatrix}(n_{1\alpha}(k-1)^\dagger \cdot n_\beta(k-1)\cdot n_{1\gamma}(0)), \tag{4.13}$$

$$(n_\alpha(k)\cdot n_\beta(k)\cdot n_\gamma(0)) = \sum_N (-)^{n_\alpha+n_\beta+n_\gamma}\Lambda(N, n_\beta)\begin{Bmatrix} n_\beta & n_{1\beta} & 1 \\ N & n_\gamma & n_\alpha \end{Bmatrix}(n_\alpha(k)\cdot n_{1\beta}(k-1)\cdot N, n_\gamma), \tag{4.14}$$

$$(n_\alpha(k)\cdot n_\beta(k-1)\cdot n_\gamma(1)) = \sum_{N(k-1)} \frac{(-)^{k-1}\delta^N_{n_\beta}\delta^{N_{k-1}}_{n(k-1)\beta}}{\Lambda(N, N_{k-1})}(n_\alpha(k)\cdot N(k-1)^\dagger\cdot n_\gamma(1)) \times (N(k-1)\cdot n_\beta(k-1)\cdot 0(0)). \tag{4.15}$$

The results (4.10), ..., (4.13) are easily derived by combining definition (2.2) with the contraction properties of 3-j coefficients, and noting formulae such as (4.9). Results (4.14) and (4.15) are not so straightforward.

The proof of (4.14) relies on combining the more basic results

$$\boldsymbol{Y}_{n_\beta(k)} = (-)^{n_\beta - m_\beta}\Lambda(n_\beta)\sum_{\substack{m_{1\beta}\\ \mu}}\begin{pmatrix} n_\beta & n_{1\beta} & 1 \\ m_\beta & -m_{1\beta} & -\mu \end{pmatrix}\boldsymbol{Y}_{n_{1\beta}(k-1)}\boldsymbol{e}_\mu,$$

$$\boldsymbol{Y}_{n_\alpha(k)}\cdot\boldsymbol{Y}_{n_{1\beta}(k-1)} = \sum_{\substack{N(1)\\ M}}(n_\alpha(k);\, m_\alpha\cdot n_{1\beta}(k-1);\, m_{1\beta}\cdot N(1);\, M)\,\overline{\boldsymbol{Y}}_{N(1)},$$

$$\overline{\boldsymbol{Y}}_{N(1)}\cdot\boldsymbol{e}_\mu = (-)^{N-M}\Lambda(N)\begin{pmatrix} N & N_1 & 1 \\ M & \mu - M & -\mu \end{pmatrix}\overline{\boldsymbol{Y}^{M-\mu}_{N_1}},$$

with the rank-0 version of (3.2) and the 3-j contraction properties. The 6-j triangle rules restrict the sum in (4.14) to at most 3 terms satisfying

$$|n_\gamma - 1| \leqslant N \leqslant n_\gamma + 1 \quad \text{and} \quad |n_\alpha - n_{1\beta}| \leqslant N \leqslant n_\alpha + n_{1\beta}.$$

The proof of (4.15) relies on the more basic results

$$(n_\alpha(k)\cdot n_\beta(k-1)\cdot n_\gamma(1)) = (n_\gamma(1)\cdot n_\alpha(k)\cdot n_\beta(k-1)),$$

$$\boldsymbol{Y}_{n_\gamma(1)}\cdot\boldsymbol{Y}_{n_\alpha(k)} = \sum_{\substack{N(k-1)\\ M}}(n_\alpha(k);\, m_\alpha\cdot N(k-1);\, M^*\cdot n_\gamma(1);\, m_\gamma)\,\boldsymbol{Y}_{N(k-1)},$$

$$\frac{1}{4\pi}\int \boldsymbol{Y}_{N(k-1)}\cdot\boldsymbol{Y}_{n_\beta(k-1)}\,\mathrm{d}\Omega = (N(k-1)\cdot n_\beta(k-1)\cdot 0(0))\begin{pmatrix} N_{k-1} & n_{(k-1)\beta} & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} N & n_\beta & 0 \\ M & m_\beta & 0 \end{pmatrix},$$

$$\begin{pmatrix} N & n_\beta & 0 \\ M & m_\beta & 0 \end{pmatrix} = (-)^{N+M}\delta^{n_\beta}_N\delta^{-m_\beta}_M/\Lambda(n_\beta),$$

and 3-j contraction properties.

The recursive relations (4.10), ..., (4.15) will be illustrated by deriving some new results for coupling integrals involving rank-2 and rank-3 harmonics. These new results will be used in the applications of § 7. Starting from the fundamental result (4.1) in the reduced form

$$(n_\alpha(0)\cdot n_\beta(0)\cdot n_\gamma(0)) = \Lambda(n_\alpha, n_\beta, n_\gamma),$$

results (4.10), ..., (4.15) imply (with the aid of 3-j and 6-j contraction properties)

$$(n_\alpha(1)\cdot n_\beta(1)\cdot n_\gamma(0)) = (-)^{n_\alpha+n_{1\alpha}}\Lambda(n_\alpha(1), n_\beta(1), n_\gamma)\begin{Bmatrix} n_\alpha & n_{1\alpha} & 1 \\ n_{1\beta} & n_\beta & n_\gamma \end{Bmatrix},$$

which is the reduced form of (4.2);

$$
\begin{aligned}
&(n_\alpha(2)\cdot n_\beta(1)\cdot n_\gamma(1))\\
&\quad = (-)^{n_\alpha+n_{1\beta}+n_\gamma}\Lambda(n_\alpha,n_\beta)\begin{Bmatrix} n_\alpha & n_{1\alpha} & 1\\ n_{1\beta} & n_\beta & n_\gamma\end{Bmatrix}(n_{1\alpha}(1)\cdot n_\gamma(1)\cdot n_{1\beta}(0)),\\
&\quad = (-)^{n_\alpha+n_{2\alpha}}\Lambda(n_\alpha(2),n_\beta(1),n_\gamma(1))\begin{pmatrix} n_\alpha & n_{1\alpha} & 1\\ n_{1\beta} & n_\beta & n_\gamma\end{pmatrix}\begin{pmatrix} n_{1\alpha} & n_{2\alpha} & 1\\ n_{1\gamma} & n_\gamma & n_{1\beta}\end{pmatrix}, \qquad (4.16)\\
&(n_\alpha(2)\cdot n_\beta(2)\cdot n_\gamma(0))\\
&\quad = \sum_N (-)^{n_\alpha+n_\beta+n_\gamma}\Lambda(N,n_\beta)\begin{Bmatrix} n_\beta & n_{1\beta} & 1\\ N & n_\gamma & n_\alpha\end{Bmatrix}(n_\alpha(2)\cdot n_{1\beta}(1)\cdot N,n_\gamma)\\
&\quad = (-)^{n_\beta+n_{2\beta}}\Lambda(n_\alpha(2),n_\beta(2),n_\gamma)\begin{Bmatrix} n_\alpha & n_{1\alpha} & 1\\ n_\gamma & n_{2\alpha} & n_{2\beta}\\ n_\beta & 1 & n_{1\beta}\end{Bmatrix},
\end{aligned}
$$

with the special case

$$
\begin{aligned}
&(n_\alpha(2)\cdot n_\beta(2)\cdot 0(0))\\
&\quad = (-)^{n_{1\alpha}+n_{1\beta}}\Lambda(n_{1\alpha},n_\beta(2))\begin{Bmatrix} n_{1\alpha} & n_\beta & 1\\ n_{1\beta} & n_{2\beta} & 1\end{Bmatrix}\delta^{n_{2\beta}}_{n_{2\alpha}}\delta^{n_\beta}_{n_\alpha}; \qquad (4.17)\\
&(n_\alpha(3)\cdot n_\beta(2)^\dagger\cdot n_\gamma(1))\\
&\quad = (-)^{n_\alpha+n_{1\beta}+n_\gamma}\Lambda(n_\alpha,n_\beta)\begin{Bmatrix} n_\alpha & n_{1\alpha} & 1\\ n_{1\beta} & n_\beta & n_\gamma\end{Bmatrix}(n_{1\alpha}(2)\cdot n_{1\beta}(1)\cdot n_\gamma(1)),\\
&\quad = (-)^{n_\alpha+n_{1\alpha}+n_{3\alpha}+n_{1\beta}+n_\gamma}\Lambda(n_\alpha(3),n_\beta(2),n_\gamma(1))\\
&\qquad\qquad \times\begin{Bmatrix} n_\alpha & n_{1\alpha} & 1\\ n_{1\beta} & n_\beta & n_\gamma\end{Bmatrix}\begin{Bmatrix} n_{1\alpha} & n_{2\alpha} & 1\\ n_{2\beta} & n_{1\beta} & n_\gamma\end{Bmatrix}\begin{Bmatrix} n_{2\alpha} & n_{3\alpha} & 1\\ n_{1\gamma} & n_\gamma & n_{2\beta}\end{Bmatrix};\\
&(n_\alpha(3)\cdot n_\beta(2)\cdot n_\gamma(1))\\
&\quad = \sum_{N(2)}\frac{\delta^N_{n_\beta}\delta^{N_2}_{n_{2\beta}}}{\Lambda(N,N_2)}(n_\alpha(3)\cdot N(2)^\dagger\cdot n_\gamma(1))\,(N(2)\cdot n_\beta(2)\cdot 0(0))\\
&\quad = (-)^{n_\alpha+n_{1\alpha}+n_{3\alpha}+n_{1\beta}+n_\gamma}\Lambda(n_\alpha(3),n_\beta(2),n_\gamma(1))\begin{Bmatrix} n_{2\alpha} & n_{3\alpha} & 1\\ n_{1\gamma} & n_\gamma & n_{2\beta}\end{Bmatrix}\begin{Bmatrix} n_\alpha & n_\beta & n_\gamma\\ n_{1\alpha} & 1 & n_{2\alpha}\\ 1 & n_{1\beta} & n_{2\beta}\end{Bmatrix}. \qquad (4.18)
\end{aligned}
$$

The preceding paragraphs show that the reduced integrals can be written in terms of special Wigner 6-*j* and 9-*j* coefficients. All these coefficients can be simply evaluated by using Table 4 and Appendices II and III of Brink & Satchler (1968), and Table 1 of James (1974). The 3-*j* coefficients with zero bottom row are also simple factors (Brink & Satchler 1968, equation (2.35)). Thus, the problem of evaluating coupling integrals reduces to one of evaluating the general 3-*j* coefficient

$$\begin{pmatrix} n_\alpha & n_\beta & n_\gamma\\ m_\alpha & m_\beta & m_\gamma\end{pmatrix}.$$

This quantity can be evaluated using standard computer subroutines based on the procedures discussed by Melvin & Swamy (1957), Wills (1971), James (1973) and Winch (1974); or using tables (Rotenberg *et al.* 1959).

Many useful properties of the coupling integrals follow directly from the standard properties of Wigner coefficients. Most important are the zero selection rules listed below.

All coupling integrals are zero unless

(i) $|m_\alpha| < n_\alpha$, etc.;

(ii) $n_\alpha, n_\beta, n_\gamma$ satisfy the triangle inequality;
$n_\alpha, n_{1\alpha}, 1$ satisfy the triangle inequality;
$n_{1\alpha}, n_{2\alpha}, 1$ satisfy the triangle inequality;
etc.;

(iii) $m_\alpha + m_\beta + m_\gamma = 0$.

$(n_\alpha(0);\, m_\alpha \cdot n_\beta(0);\, m_\beta \cdot n_\gamma(0);\, m_\gamma)$ is zero unless

(i) $n_\alpha + n_\beta + n_\gamma$ is even.

$(n_\alpha(1);\, m_\alpha \cdot n_\beta(1);\, m_\beta \cdot n_\gamma(0);\, m_\gamma)$ is zero unless

(i) $n_{1\alpha} + n_{1\beta} + n_\gamma$ is even;

(ii) $n_{1\alpha}, n_{1\beta}, n_\gamma$ satisfy the triangle inequality.

$(n_\alpha(1);\, m_\alpha \times n_\beta(1);\, m_\beta \cdot n_\gamma(1);\, m_\gamma)$ is zero unless

(i) $n_{1\alpha} + n_{1\beta} + n_{1\gamma}$ is even;

(ii) $n_{1\alpha}, n_{1\beta}, n_{1\gamma}$ satisfy the triangle inequality.

$(n_\alpha(2);\, m_\alpha \cdot n_\beta(1);\, m_\beta \cdot n_\gamma(1);\, m_\gamma)$ is zero unless

(i) $n_{2\alpha} + n_{1\beta} + n_{1\gamma}$ is even;

(ii) $n_{2\alpha}, n_{1\beta}, n_{1\gamma}$ satisfy the triangle inequality;

(iii) $n_{1\alpha}, n_{1\beta}, n_\gamma$ satisfy the triangle inequality.

$(n_\alpha(2);\, m_\alpha \cdot n_\beta(2);\, m_\beta \cdot n_\gamma(0);\, m_\gamma)$ is zero unless

(i) $n_{2\alpha} + n_{2\beta} + n_\gamma$ is even;

(ii) $n_{2\alpha}, n_{2\beta}, n_\gamma$ satisfy the triangle inequality.

$(n_\alpha(3);\, m_\alpha \cdot n_\beta(2);\, m_\beta{}^{\dagger} \cdot n_\gamma(1);\, m_\gamma)$ is zero unless

(i) $n_{3\alpha} + n_{2\beta} + n_{1\gamma}$ is even;

(ii) $n_{3\alpha}, n_{2\beta}, n_{1\gamma}$ satisfy the triangle inequality;

(iii) $n_{1\alpha}, n_{1\beta}, n_\gamma$ satisfy the triangle inequality;

(iv) $n_{2\alpha}, n_{2\beta}, n_\gamma$ satisfy the triangle inequality.

$(n_\alpha(3);\, m_\alpha \cdot n_\beta(2);\, m_\beta \cdot n_\gamma(1);\, m_\gamma)$ is zero unless

(i) $n_{3\alpha} + n_{2\beta} + n_{1\gamma}$ is even;

(ii) $n_{3\alpha}, n_{2\beta}, n_{1\gamma}$ satisfy the triangle inequality;

(iii) $n_{2\alpha}, n_{2\beta}, n_\gamma$ satisfy the triangle inequality.

Such selection rules greatly reduce the number of coupling integrals needed in any given problem. The numerous symmetry properties of Wigner coefficients also lead to symmetry properties for coupling integrals, and on occasions these symmetries imply additional selection rules. Details are left as exercises for the reader, but see James (1973, 1974) who discusses some implications of Wigner symmetries for the integrals (4.1) and (4.3).

5. Symmetric and antisymmetric rank-2 tensors

In practice, one often deals with rank-2 tensors which are purely symmetric or antisymmetric, or whose trace, symmetric and antisymmetric parts have particular physical significance. Thus, it is desirable to be able to easily expand symmetric or antisymmetric tensors, and to be able to

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easily extract the symmetric and antisymmetric parts of a general rank-2 tensor. This section shows that these aims are readily fulfilled by relating the rank-2 tensor-harmonics defined by (2.2) to the tensor-harmonics defined by Zerilli (1970).

The general rank-2 tensor harmonic may be decomposed into trace ($\boldsymbol{T}$), antisymmetric ($\boldsymbol{A}$) and trace-free symmetric ($\boldsymbol{S}$) parts according to the formula

$$\boldsymbol{Y}_{n(2)} = \boldsymbol{T}+\boldsymbol{A}+\boldsymbol{S},$$

where
$$\boldsymbol{T} = \frac{\boldsymbol{\delta}}{3}\times\{\text{trace of } \boldsymbol{Y}_{n(2)}\},$$

$$\boldsymbol{S} = \tfrac{1}{2}(\boldsymbol{Y}_{n(2)}+\boldsymbol{Y}_{n(2)}{}^{\dagger})-\boldsymbol{T}, \tag{5.1}$$

$$\boldsymbol{A} = \tfrac{1}{2}(\boldsymbol{Y}_{n(2)}-\boldsymbol{Y}_{n(2)}{}^{\dagger}). \tag{5.2}$$

Using (2.3), with $k = 2$, and the orthogonality of 3-j coefficients, one finds

$$\boldsymbol{T} = (-)^{n+n_1}\frac{\boldsymbol{\delta}}{3}\frac{\Lambda(n_1)}{\Lambda(n)}\,Y_n^m\,\delta^n_{n_2}. \tag{5.3}$$

Therefore $\boldsymbol{Y}_{n(2)}$ is trace-free unless $n_2 = n$. Evaluation of the integral

$$\frac{1}{4\pi}\int \boldsymbol{Y}_{N(2)}{}^{*}\cdot Y_n^m\boldsymbol{\delta}\,\mathrm{d}\Omega$$

allows $\boldsymbol{T}$ to be rewritten as

$$\boldsymbol{T} = \delta^n_{n_2}\sum_{N_1}\frac{(-)^{n_1+N_1}\Lambda(n_1,N_1)}{3(2n+1)}\,\boldsymbol{Y}_{n,N_1,n}. \tag{5.4}$$

Expansion of $\boldsymbol{A}$ and $\boldsymbol{S}$ in terms of the $\boldsymbol{Y}$-harmonics awaits the corresponding expansion of $\boldsymbol{Y}_{n(2)}{}^{\dagger}$. But

$$\boldsymbol{Y}_{n(2)}{}^{\dagger} = \sum_{\substack{N(2)\\M}}(N(2);\,M^{*}\cdot n(2);\,m^{\dagger}\cdot 0(0);\,0)\,\boldsymbol{Y}_{N(2)},$$

which (4.14) reduces to

$$\boldsymbol{Y}_{n(2)}{}^{\dagger} = \sum_{N_1}(-)^{n_1+N_1}\Lambda(n_1,N_1)\begin{Bmatrix} n & n_1 & 1\\ n_2 & N_1 & 1\end{Bmatrix}\boldsymbol{Y}_{n,N_1,n_2}. \tag{5.5}$$

By substituting (5.5) into (5.1) and (5.2), and choosing appropriate 6-j coefficients from Table 4 of Brink & Satchler (1968), one finds (as expected) that corresponding to the 9 rank-2 tensors $\boldsymbol{Y}_{n(2)}$ ($n_1 = n, n\pm 1$; $n_2 = n_1, n_1\pm 1$), there are 5 linearly independent trace-free symmetric tensors and 3 antisymmetric tensors. These tensors are given explicitly in appendix B. Together with the trace, these are precisely the tensor harmonics defined by Zerilli (1970), who extended the definition of symmetric rank-2 tensors given by Mathews (1962). In slightly different notation, Zerilli defined tensor harmonics

$$\boldsymbol{T}^{(l)}_{n,n_2;m} = (-)^{n-m}\Lambda(n)\sum_{m_2,\mu}\begin{pmatrix} n & n_2 & l\\ m & -m_2 & -\mu\end{pmatrix}Y^{m_2}_{n_2}\boldsymbol{t}^{(l)}_{\mu}, \tag{5.6}$$

where
$$\boldsymbol{t}^{(l)}_{\mu} = (-)^{\mu}\Lambda(l)\sum_{\mu_1,\mu_2}\begin{pmatrix} l & 1 & 1\\ \mu & -\mu_1 & -\mu_2\end{pmatrix}\boldsymbol{e}_{\mu_1}\boldsymbol{e}_{\mu_2}\quad\text{and}\quad l = 0,\,1,\,2.$$

Zerilli's tensors are a complete orthogonal set for expanding rank-2 tensor functions on a sphere; and for $l = 0, 1, 2$, correspond respectively to trace, antisymmetric and trace-free symmetric components. Since definitions (5.6) and (2.2) represent alternative Clebsch–Gordan couplings of the same irreducible spherical tensors $Y_{n_2}^{m_2}$, $\boldsymbol{e}_{\mu_1}$, $\boldsymbol{e}_{\mu_2}$, the Mathews–Zerilli tensors are related to the $\boldsymbol{Y}$'s via the definition of 6-j coefficients. Specifically, evaluation of the integral

$$\int \boldsymbol{Y}_{N(2)}{}^{*} \cdot \boldsymbol{T}^{(l)}_{n,\,n_2;\,m}\,\mathrm{d}\Omega,$$

shows that

$$\boldsymbol{T}^{(l)}_{n,\,n_2;\,m} = \sum_{n_1} (-)^{n+n_2+l} \Lambda(n_1, l) \begin{Bmatrix} 1 & l & 1 \\ n & n_1 & n_2 \end{Bmatrix} \boldsymbol{Y}_{n(2)}, \tag{5.7}$$

or, since this is an orthogonal transformation,

$$\boldsymbol{Y}_{n(2)} = \sum_{l} (-)^{n+n_2+l} \Lambda(n_1, l) \begin{Bmatrix} 1 & l & 1 \\ n & n_1 & n_2 \end{Bmatrix} \boldsymbol{T}^{(l)}_{n,\,n_2;\,m}. \tag{5.8}$$

Use of (5.7) and (5.8) immediately gives $\boldsymbol{T}$ ($l = 0$), $\boldsymbol{A}$ ($l = 1$) and $\boldsymbol{S}$ ($l = 2$):

$$\boldsymbol{T} = (-)^{n+n_2} \Lambda(n_1) \begin{Bmatrix} 1 & 0 & 1 \\ n & n_1 & n_2 \end{Bmatrix} \boldsymbol{T}^{(0)}_{n,\,n_2,\,m}, \tag{5.9}$$

$$= \sum_{N_1} \Lambda(n_1, N_1) \begin{Bmatrix} 1 & 0 & 1 \\ n & n_1 & n_2 \end{Bmatrix} \begin{Bmatrix} 1 & 0 & 1 \\ n & N_1 & n_2 \end{Bmatrix} \boldsymbol{Y}_{n,\,N_1,\,n_2}, \tag{5.10}$$

which is equivalent to (5.3) and (5.4);

$$\boldsymbol{A} = (-)^{n+n_2+1} \Lambda(n_1, 1) \begin{Bmatrix} 1 & 1 & 1 \\ n & n_1 & n_2 \end{Bmatrix} \boldsymbol{T}^{(1)}_{n,\,n_2,\,m},$$

$$= \sum_{N_1} 3\Lambda(n_1, N_1) \begin{Bmatrix} 1 & 1 & 1 \\ n & n_1 & n_2 \end{Bmatrix} \begin{Bmatrix} 1 & 1 & 1 \\ n & N_1 & n_2 \end{Bmatrix} \boldsymbol{Y}_{n,\,N_1,\,n_2}; \tag{5.11}$$

$$\boldsymbol{S} = (-)^{n+n_2} \Lambda(n_1, 2) \begin{Bmatrix} 1 & 2 & 1 \\ n & n_1 & n_2 \end{Bmatrix} \boldsymbol{T}^{(2)}_{n,\,n_2,\,m}$$

$$= \sum_{N_1} 5\Lambda(n_1, N_1) \begin{Bmatrix} 1 & 2 & 1 \\ n & n_1 & n_2 \end{Bmatrix} \begin{Bmatrix} 1 & 2 & 1 \\ n & N_1 & n_2 \end{Bmatrix} \boldsymbol{Y}_{n,\,N_1,\,n_2}. \tag{5.12}$$

The relevant 6-j coefficients are simple factors obtainable from Table 4 of Brink & Satchler (1968). Triangle rules for these 6-j coefficients restrict the sums in (5.10), (5.11), (5.12) to at most 3, and usually less, terms (appendix B).

The ease with which one can transfer between the Zerilli harmonics and the $\boldsymbol{Y}$-harmonics via (5.7) and (5.8) makes the Zerilli harmonics a very useful supplement to the method of this paper. For example, when working with a symmetric tensor, one can first expand it in terms of $\boldsymbol{T}^{(l)}_{n,\,n_2,\,m}$ ($l = 0$ and 2), then rewrite it in terms of the $\boldsymbol{Y}$-harmonics using (5.7), so that the properties of § 3 are applicable. Alternatively, when working with a tensor in the $\boldsymbol{Y}$-formalism, (5.10), (5.11) and (5.12) immediately give its trace, antisymmetric and trace-free components. Example (c) of § 7 illustrates this intermixing of methods explicitly.

6. Tensor harmonics defined by other authors

In addition to the rank-2 Zerilli harmonics discussed in § 5, spherical harmonic representations of tensor fields have been considered by Gel'fand & Shapiro (1956), Regge & Wheeler (1957), Backus (1966, 1967), Burridge (1966) and Phinney & Burridge (1973). The Regge–Wheeler tensors are only rank-2, and not all mutually orthogonal. Backus' tensor-representation theorem is also only rank-2, although not restricted to spheres. The Regge–Wheeler and Backus tensors

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are essentially the result of applying the operators $\boldsymbol{e}_r$, $\boldsymbol{L}$, ∇ to the scalar harmonic Y_n^m. Therefore, these tensors can be expressed in the $\boldsymbol{Y}$-formalism using properties from § 3. Zerilli has expressed the Regge–Wheeler tensors in his formalism and a simple relation exists (Burridge 1966) between Backus' theorem and some of the results of Burridge whose method is considered below.

The only general alternative to the method described herein appears to be the method of Gel'fand & Shapiro (1956) extended for practical use by Burridge (1966). In the Gel'fand–Shapiro–Burridge approach, the polar components of a rank-k tensor-function $\boldsymbol{F}_{(k)}(\theta, \phi)$ are expanded in series of generalized spherical harmonics. In place of (3.3), Burridge writes (in slightly different notation)

$$\boldsymbol{F}_{(k)}(\theta, \phi) = \sum_{\substack{\mu_1, \ldots, \mu_k \\ n, m}} F_n^{\mu_1, \ldots, \mu_k; m} d^n_{m, M}(\theta)\, \mathrm{e}^{\mathrm{i}m\phi}\, \boldsymbol{e}'_{\mu_k} \ldots \boldsymbol{e}'_{\mu_1}, \tag{6.1}$$

where $M = \mu_1 + \ldots + \mu_k$, and $\boldsymbol{e}'_\mu$ ($\mu = 0, \pm 1$) are the complex spherical-polar basis vectors. Such an expansion bears little resemblance to (3.3), but comparison of (6.1) with (3.19) shows that

$$F_n^{\mu_1, \ldots, \mu_k; m} = \sum_{n_1, \ldots, n_k} K' F_{n(k)}, \tag{6.2}$$

where K' is as in (3.20). Because of 3-j orthogonality, (6.2) inverts to

$$F_{n(k)} = \sum_{\mu_1, \ldots, \mu_k} K' F_n^{\mu_1, \ldots, \mu_k; m} / (2n+1).$$

When $k = 1$, (6.2) simplifies to

$$F_n^{-1; m} = -\left[\left(\frac{n+1}{2}\right)^{\frac{1}{2}} F_{n, n-1} + \left(\frac{2n+1}{2}\right)^{\frac{1}{2}} F_{n, n} + \left(\frac{n}{2}\right)^{\frac{1}{2}} F_{n, n+1}\right],$$

$$F_n^{0; m} = n^{\frac{1}{2}} F_{n, n-1} - (n+1)^{\frac{1}{2}} F_{n, n+1},$$

$$F_n^{1; m} = -\left[\left(\frac{n+1}{2}\right)^{\frac{1}{2}} F_{n, n-1} - \left(\frac{2n+1}{2}\right)^{\frac{1}{2}} F_{n, n} + \left(\frac{n}{2}\right)^{\frac{1}{2}} F_{n, n+1}\right].$$

A minor disadvantage of the Burridge approach is apparent here in the mixing of the toroidal term $F_{n, n}$ with the non-toroidal $F_{n, n\pm 1}$. This point manifests itself again in § 7 (*c*). The main advantage of the new method based on definition (2.2) is that the numerous known properties of scalar and vector spherical harmonics are readily generalized as shown in § 3. Moreover, the complexity of the generalized properties does not increase with rank. Compare, for example, the Gradient Formula (3.5), containing at most two terms, with the corresponding formula (11.1) of Burridge (1966), the latter containing about k terms in the rank-k case. In addition, as shown in § 4, definition (2.2) allows systematic calculation of coupling integrals required for spectral analysis of products of tensors. The Gel'fand–Shapiro–Burridge approach has apparently only been applied to linear terms (Phinney & Burridge 1973), although use of (4.4) would permit extension to nonlinear terms. In this regard, a slight numerical advantage of the Burridge formalism will be seen later in § 8. That is, computation of the Burridge coefficient $F_n^{\mu_1, \ldots, \mu_k; m}$ for a nonlinear interaction, requires marginally less arithmetic operations than computation of the corresponding $\boldsymbol{Y}$-coefficient $F_{n(k)}$.

7. Applications to partial differential equations

The main application of the new tensor spherical harmonics is in deriving the spectral form of partial differential equations in spherical coordinates. This section begins with two simple applications to the tensor Laplace and Helmholtz equations. A third application to the equations of free elastic vibrations illustrates the advantage of supplementing the method with the rank-2 harmonics of Zerilli (1970), and allows comparison with the work of Phinney & Burridge (1973). Finally, the theory is applied to equations from fluid mechanics and magnetofluidmechanics. The ease with which the nonlinear terms in these equations can be analysed is a feature of the new method.

(a) *The tensor Laplace equation*

Suppose it is required to find the general solution of

$$\nabla^2 \boldsymbol{L}_{(k)} = 0, \tag{7.1}$$

where $\boldsymbol{L}_{(k)}(r, \theta, \phi)$ is a rank-k tensor function, $k \geqslant 0$. Analogous to (3.3) we expand

$$\boldsymbol{L}_{(k)} = \sum_{\substack{n(k)\\m}} L_{n(k)} \boldsymbol{Y}_{n(k)},$$

and employ property (3.15) to find that the spectral form of (7.1) is

$$D_{n_k} L_{n(k)} = 0,$$

with solutions $\qquad L_{n(k)} = r^{n_k} \quad \text{and} \quad r^{-n_k-1}.$

Thus, the general solution of (7.1) is

$$\boldsymbol{L}_{(k)} = \sum_{\substack{n(k)\\m}} [E_{n(k)} r^{n_k} + I_{n(k)} r^{-n_k-1}] \boldsymbol{Y}_{n(k)}, \tag{7.2}$$

where E and I are independent of r, θ, ϕ. Note that such solutions can be made solenoibal by using (3.10). For example, consider a rank-$(k+1)$ solution, $\boldsymbol{L}_{(k+1)}$, obtained by replacing k in (7.2) by $k+1$. Equation (3.10) implies that $\boldsymbol{L}_{(k+1)}$ will be solenoidal provided

$$I_{n(k),\, n_k-1} = E_{n(k),\, n_k+1} = 0$$

for all $n(k)$, m.

(b) *The tensor Helmholtz equation*

Suppose it is required to solve

$$\nabla^2 \boldsymbol{H}_{(k)} + p^2 \boldsymbol{H}_{(k)} = 0, \quad p \neq 0, \tag{7.3}$$

where $\boldsymbol{H}_{(k)}(r, \theta, \phi)$ is a rank-k tensor function. Proceeding as in (a), the spectral form of (7.3) is

$$(D_{n_k} + p^2) H_{n(k)} = 0,$$

with solutions $\qquad H_{n(k)} = j_{n_k}(pr) \quad \text{and} \quad y_{n_k}(pr),$

where j and y are spherical Bessel functions. The general solution of (7.3) is thus

$$\boldsymbol{H}_{(k)} = \sum_{\substack{n(k)\\m}} [E_{n(k)} j_{n_k}(pr) + I_{n(k)} y_{n_k}(pr)] \boldsymbol{Y}_{n(k)}, \tag{7.4}$$

where E and I are independent of r, θ, ϕ.

Equation (3.10) and the standard recurrence relations (Antosiewicz 1965)

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{1-n}{r}\right) \begin{Bmatrix} j_{n-1}(pr) \\ y_{n-1}(pr) \end{Bmatrix} = -p \begin{Bmatrix} j_n(pr) \\ y_n(pr) \end{Bmatrix},$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{n+2}{r}\right) \begin{Bmatrix} j_{n+1}(pr) \\ y_{n+1}(pr) \end{Bmatrix} = p \begin{Bmatrix} j_n(pr) \\ y_n(pr) \end{Bmatrix},$$

imply that a rank-$(k+1)$ solution $\boldsymbol{H}_{(k+1)}$, of type (7.4), will be solenoidal if

$$n_k^{\frac{1}{2}} E_{n(k),\, n_k-1} = -(n_k+1)^{\frac{1}{2}} E_{n(k),\, n_k+1},$$
$$n_k^{\frac{1}{2}} I_{n(k),\, n_k-1} = -(n_k+1)^{\frac{1}{2}} I_{n(k),\, n_k+1}.$$

(*c*) *Free elastic vibrations*

The method of this paper applies to any of the equations describing elastic vibrations in a sphere; for example, those used to model Earth vibrations (Dahlen 1968). For conciseness we restrict ourselves here to a simple illustrative subset of equations, namely the Fourier-transformed momentum and elastic-constitutive equations considered by Phinney & Burridge (1973):

$$-\rho\omega^2 \boldsymbol{u} = \nabla\cdot\boldsymbol{\tau}, \tag{7.5}$$

$$\boldsymbol{\tau} = \lambda(\nabla\cdot\boldsymbol{u})\,\boldsymbol{\delta} + \mu(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\dagger}). \tag{7.6}$$

Here, ρ is density, assumed constant; $\boldsymbol{\tau}$ is elastic stress, $\boldsymbol{u}$ displacement, ω angular frequency, μ and λ Lamé constants, and $\boldsymbol{\delta}$ is the unit dyadic in (2.6).

Analogous to (3.3) we expand

$$\boldsymbol{u} = \sum_{\substack{n(1)\\ m}} u_{n(1)} \boldsymbol{Y}_{n(1)}.$$

However, since $\boldsymbol{\tau}$ is symmetric, the number of spectral equations can be reduced by first expanding $\boldsymbol{\tau}$ in Zerilli tensors. Thus, we write

$$\boldsymbol{\tau} = \sum_{\substack{n,\,n_2\\ m,\,l}} \tau^{(l)}_{n,\,n_2,\,m} \boldsymbol{T}^{(l)}_{n,\,n_2,\,m},$$

where l takes the values 0 and 2 only, representing trace and trace-free symmetric parts. To find $\nabla\cdot\boldsymbol{\tau}$ we transform to the $\boldsymbol{Y}$-formalism using (5.7):

$$\boldsymbol{\tau} = \sum_{\substack{n(2)\\ m,\,l}} (-)^{n+n_2} \varLambda(n_1, l) \begin{Bmatrix} 1 & l & 1 \\ n & n_1 & n_2 \end{Bmatrix} \tau^{(l)}_{n,\,n_2,\,m} \boldsymbol{Y}_{n(2)}.$$

The spectral form of (7.5) then follows directly from the divergence property (3.10):

$$-\rho\omega^2 u_{n(1)} = \sum_{n_2,\,l} (-)^{n+n_2} \varLambda(n_1, l) \begin{Bmatrix} 1 & l & 1 \\ n & n_1 & n_2 \end{Bmatrix} G(n_1, n_2)\, \partial^{n_1}_{n_2} \tau^{(l)}_{n,\,n_2,\,m}. \tag{7.7}$$

To find the spectral form of (7.6), it is convenient to rewrite (7.6) as

$$\boldsymbol{\tau} = (3\lambda + 2\mu)\, \boldsymbol{T}(\nabla\boldsymbol{u}) + 2\mu \boldsymbol{S}(\nabla\boldsymbol{u}), \tag{7.8}$$

where $\boldsymbol{T}$ and $\boldsymbol{S}$ represent the trace and trace-free symmetric parts of $\nabla\boldsymbol{u}$. The gradient formula (3.5) and the orthogonal $\boldsymbol{Y}$–$\boldsymbol{T}$ transformation (5.8) combine to give the required spectral form of (7.8):

$$\tau^{(l)}_{n,\,n_2,\,m} = \lambda' \sum_{n_1} (-)^{n+n_2} \varLambda(n_1, l) \begin{Bmatrix} 1 & l & 1 \\ n & n_1 & n_2 \end{Bmatrix} G(n_1, n_2)\, \partial^{n_2}_{n_1} u_{n(1)}, \tag{7.9}$$

where
$$\lambda' = \begin{cases} 3\lambda + 2\mu, & \text{if } l = 0, \\ 2\mu, & \text{if } l = 2. \end{cases}$$

For fixed n and m, the spectral equations (7.7) and (7.9) generally represent nine equations in the nine unknowns $u^m_{n,\,n_1}$ $(n_1 = n, n\pm 1)$, $\tau^{(l)}_{n,\,n_2,\,m}$ $(n_2 = n$ if $l = 0$; $n_2 = n, n\pm 1, n\pm 2$ if $l = 2)$.

Wigner-coefficient selection rules restrict the sums in (7.7) and (7.9) to at most three (and usually less) terms, and ensure that the equations containing the toroidal displacement coefficients $u^m_{n,n}$ are uncoupled from the equations containing the non-toroidal coefficients $u_{n,n\pm 1}$. More explicitly, suppose $n_1 = n$ in (7.7) or (7.9). Then (3.6) requires $n_2 = n \pm 1$; and the 6-j triangle inequality $|n-l| \leqslant n_2 \leqslant n+l$ then requires $l = 2$. Conversely, suppose $n_2 = n\pm 1$ and $l = 2$ in (7.7) or (7.9). Equation (3.6) requires $n_1 = n-2$, n or $n+2$; the triangle inequality $|n-1| \leqslant n_1 \leqslant n+1$ then ensures $n_1 = n$. Thus, the equations containing $u_{n,n}$, $\tau^{(2)}_{n,n\pm 1,m}$ are uncoupled from those containing $u_{n,n\pm 1}$, $\tau^{(l)}_{n,n_2,m}$ ($n_2 = n, n\pm 2$; $l = 0, 2$). The Wigner-coefficient constants in (7.7) and (7.9) are identical and easily evaluated using (3.6), and Table 4 of Brink & Satchler (1968). Equations (7.7) and (7.9) correspond to the nine equations labelled (3.11 *a–c*) and (3.15 *a–f*) by Burridge & Phinney (1973), although toroidal–non-toroidal decoupling is not automatic in the Burridge–Phinney approach.

(*d*) *Navier–Stokes and Euler equations of motion*

Euler's equation of motion for an inviscid fluid moving in a rotating reference frame is

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + 2\boldsymbol{\Omega}\times\boldsymbol{v}\right) = -\nabla p + \rho \boldsymbol{F}. \tag{7.10}$$

Here, ρ is fluid density, $\boldsymbol{v}$ velocity, p pressure and $\boldsymbol{F}$ body force. The angular velocity $\boldsymbol{\Omega}$ of the rotating frame will be assumed constant in the z-direction, so that by (2.5), its $\boldsymbol{Y}$-expansion is simply

$$\boldsymbol{\Omega} = \Omega \boldsymbol{Y}^0_{1,0}.$$

If ρ is constant, or spherically symmetric, then use of the gradient formula (3.5) and the coupling integrals (4.3) and (4.16) allows the spectral form of (7.10) to be written immediately as

$$\rho\left(\frac{\partial v_{n(1)}}{\partial t} + V_{n(1)} - F_{n(1)}\right) = -G(n, n_1)\,\partial^{n_1}_{n}\,p_{n(0)}, \tag{7.11}$$

where

$$V_{n(1)} = \sum_{\alpha,\beta} (n_\alpha(1);\, m_\alpha\cdot n_\beta(2);\, m_\beta\cdot \overline{n(1);\, m})\, v_{n_\alpha(1)}\, G(n_{1\beta}, n_{2\beta})\, \partial^{n_{2\beta}}_{n_{1\beta}} v_{n_\beta(1)}$$
$$+\, 2\Omega \sum_{\alpha} (1,\,0;\,0\times n_\alpha(1);\, m_\alpha\cdot\overline{n(1);\,m})\, v_{n_\alpha(1)}. \tag{7.12}$$

The summations in (7.12) are over all indices with subscripts α and β. For the special case in (7.12), the cross-product coupling integral (4.3) simplifies to

$$(1,\,0;\,0\times n_\alpha(1);\, m_\alpha\cdot\overline{n(1);\,m}) = (-)^{n_1+m+1}\,6^{\frac{1}{2}}\,\mathrm{i}\Lambda(n_\alpha, n)\begin{Bmatrix} n_\alpha & n & 1\\ 1 & 1 & n_1\end{Bmatrix}\begin{Bmatrix} 1 & n_\alpha & n\\ 0 & m & -m\end{Bmatrix}\delta^{n_{1\alpha}}_{n_1}\,\delta^{m_\alpha}_{m},$$

so that the second summation in (7.12) simplifies to a sum over n_α only. Moreover, this sum contains at most three terms satisfying the triangle inequality $|n-1| \leqslant n_\alpha \leqslant n+1$. If ρ depends on θ and ϕ, then the left hand side of (7.11) is merely replaced by

$$\sum_{\alpha,\beta} \rho_{n_\alpha}\left(\frac{\partial v_{n_\beta(1)}}{\partial t} + V_{n_\beta(1)} - F_{n_\beta(1)}\right)(n_\alpha(0);\, m_\alpha\cdot n_\beta(1);\, m_\beta\cdot\overline{n(1);\,m}).$$

The Navier–Stokes equation differs from (7.10) by the addition of viscous terms proportional to $\nabla^2\boldsymbol{v}$ and $\nabla(\nabla\cdot\boldsymbol{v})$. The spectral coefficients of these terms are easily obtained from (3.15) and (3.16). Another common addition for hydromagnetic studies is a Lorentz force proportional to $(\nabla\times\boldsymbol{B})\times\boldsymbol{B}$, where $\boldsymbol{B}$ is a magnetic induction field permeating the fluid. Such a force is readily

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included by using the curl formula (3.14) and the cross-product coupling integral (4.3). Other authors have apparently only considered special simplifying cases of the Navier–Stokes equation. For example, Merilees (1968) considers the radial component of the vorticity equation with hydrostatic equilibrium in the radial direction; Frazer (1974) considers incompressible flow linearized for slow motions. Similarly, Backus (1967) linearizes the elastic equations of motion.

(*e*) *Magnetic induction equations*

The magnetic induction equation describing the interaction between a magnetic field $\boldsymbol{B}$ and a conducting fluid moving with velocity $\boldsymbol{v}$, is

$$\partial \boldsymbol{B}/\partial t = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}, \tag{7.13}$$

where η is the magnetic diffusivity. The spectral form of (7.13) is best constructed using the coupling integral (4.3) and has been discussed in detail by James (1974). We merely note here that use of the new curl result (3.14) allows the three spectral equations labelled (18), (19), (20) by James (1974) to be written together as

$$\left(\frac{\partial}{\partial t} - \eta D_{n_1}\right) B_{n(1)} = \sum_{n_1'} C(n, n_1, n_1')\, \partial_{n_1'}^{n_1} E,$$

where the induced electric field coefficient is

$$E = \sum_{\alpha, \beta} (n_\alpha(1)\,;\, m_\alpha \times n_\beta(1)\,;\, m_\beta \cdot \overline{n, n_1'\,;\, m})\, v_{n_\alpha(1)} B_{n_\beta(1)}.$$

The present paper is more concerned with higher rank tensors. For example, in mean field electrodynamics, the induction equation (7.13) can take the form (Roberts 1971)

$$\partial \boldsymbol{B}/\partial t = \nabla \times (\boldsymbol{a}_{(2)} \cdot \boldsymbol{B} + \boldsymbol{b}_{(3)} \cdot \nabla \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}. \tag{7.14}$$

Here $\boldsymbol{a}_{(2)}$, the so-called 'α-effect', and $\boldsymbol{b}_{(3)}$ are rank-2 and rank-3 tensor functions which represent the interaction of small-scale velocity and magnetic fields to produce a contribution to the larger scale field $\boldsymbol{B}$. Several authors (see, for example, Roberts 1972; Stix 1971) have derived the spectral form of (7.14) for various special cases. The spectral form of the general equation (7.14) is easily obtained in the $\boldsymbol{Y}$-formalism. One simply combines the gradient, Laplacian and curl results (3.5), (3.15) and (3.14) with the coupling integrals (4.16) and (4.18), to obtain

$$\left(\frac{\partial}{\partial t} - \eta D_{n_1}\right) B_{n(1)} = \sum_{n_1'} C(n, n_1, n_1')\, \partial_{n_1'}^{n_1} [E_a + E_b].$$

Here the induced electric field coefficients are

$$E_a = \sum_{\alpha, \beta} (n_\alpha(2)\,;\, m_\alpha \cdot n_\beta(1)\,;\, m_\beta \cdot \overline{n, n_1'\,;\, m})\, a_{n_\alpha(2)} B_{n_\beta(1)},$$

$$E_b = \sum_{\alpha, \beta} (n_\alpha(3)\,;\, m_\alpha \cdot n_\beta(2)\,;\, m_\beta \cdot \overline{n, n_1'\,;\, m})\, b_{n_\alpha(3)} G(n_{1\beta}, n_{2\beta})\, \partial_{n_{1\beta}}^{n_{2\beta}} B_{n_\beta(1)}.$$

8. Efficient computation of nonlinear terms

There are essentially two different types of spectral analysis problems which arise in practice. The first type is linear in that it contains, at worst, products of known functions and only one unknown function. A common example is the so-called kinematic dynamo problem of geomagnetism, where the induction equation (7.13) is solved for the magnetic field $\boldsymbol{B}$, assuming the velocity $\boldsymbol{v}$ is known. Spectral analysis and finite differencing reduces such problems to linear eigenvalue problems. The governing matrices are readily computed by using coupling integrals such as those described in § 4, and this computation occupies an insignificant portion of the total computing time.

The second type of spectral problem concerns nonlinear equations which contain products of two or more unknown functions. For example, the full solution of the geomagnetic dynamo problem requires that $\boldsymbol{v}$ not be prescribed, but that the induction equation be coupled with the Navier–Stokes, continuity and thermodynamic equations. Computation of the spectral coefficients of the nonlinear terms may then require a formidable amount of time and storage, especially since such computation must be repeated at each point of a time/radius finite-difference grid. Orszag (1970) has shown how such time and storage requirements may be significantly reduced, and we will now show how Orszag's method generalizes for tensor-harmonics.

Orszag's method (originally devised for the vertical component of the atmospheric vorticity equation), relies in part on use of the Fast Fourier Transform algorithm. The applicability of this algorithm to tensor-harmonics relies on equation (3.19), which allows the ϕ-dependence of $\boldsymbol{Y}_{n(k)}\cdot\boldsymbol{e}'_{\mu_1}\ldots\boldsymbol{e}'_{\mu_k}$ to be isolated in the single term $e^{im\phi}$. The generalization of Orszag's method will be illustrated by considering the exterior product of two vectors $\boldsymbol{a}_{(1)}$, $\boldsymbol{b}_{(1)}$, to form a dyadic. This example illustrates the approach to the general coupling problem, which is only notationally more complex. The most commonly occurring cases are simpler. Consider then the product

$$\boldsymbol{a}_{(1)}\boldsymbol{b}_{(1)} = \sum_{n(2),m} x_{n(2)}\boldsymbol{Y}_{n(2)}, \tag{8.1}$$

where

$$\begin{aligned} x_{n(2)} &= \sum_{\alpha,\beta}(n(2);\, m^*\cdot n_\alpha(1);\, m_\alpha\cdot n_\beta(1);\, m_\beta)\, a_{n_\alpha(1)}b_{n_\beta(1)} \\ &= \sum_{\alpha,\beta}\overline{(n(2);\, m\cdot n_\beta(1)};\, m_\beta\cdot n_\alpha(1);\, m_\alpha)\, a_{n_\alpha(1)}b_{n_\beta(1)}. \end{aligned} \tag{8.2}$$

The problem is to compute the spectral coefficients $x_{n(2)}$ ($n = 0, 1, \ldots; |m| \leqslant n; |n-1| \leqslant n_1 \leqslant n+1$; $|n_1-1| \leqslant n_2 \leqslant n_1+1$), given $a_{n(1)}$ and $b_{n(1)}$.

In practice, finite truncation levels must be introduced for the range of n_α, n_β. For simplicity, suppose $n_\alpha = 0, \ldots, N$; $n_\beta = 0, \ldots, N$. Selection rules for the integral (4.16) then imply that the spectral coefficients $x_{n(2)}$ exist in the range $n = 0, \ldots, 2N$; although they would usually only be required for $n \leqslant N$. Further, from (3.4),

$$x^{-m}_{n(2)} = (-)^{n+n_2+m}\overline{x^m_{n(2)}}, \tag{8.3}$$

so that it is sufficient to consider the range $0 \leqslant m \leqslant n$.

Assume firstly that we compute $x_{n(2)}$ using (8.2). The number of multiplications required by each parameter is (i) at most 3 for each of $n_{1\alpha}$ and $n_{1\beta}$; (ii) $n+1$ for m, with n ranging from 0 to N; (iii) $2n_\alpha+1$ for m_α, with n_α ranging from 0 to N; (iv) at most $2n_\alpha+1$ for n_β, corresponding to the range $|n-n_\alpha|$ to $n+n_\alpha$. Selection rules for 3-j coefficients fix $m_\beta = m-m_\alpha$. This gives a total of

$$3^2\sum_{n=0}^{N}(n+1)\sum_{n_\alpha=0}^{N}(2n_\alpha+1)^2 \sim 6N^5,$$

real × complex × complex multiplications, each of which is equivalent to 6 real multiplications. Selection rules for 3-j coefficients with zero bottom row – see § 4 – require $n_{1\alpha}+n_{1\beta}+n_2$ to be even, reducing the number of terms in (8.2) by a factor of 2. Thus the total number of real multiplications is about $18N^5$. For each 6 real multiplications, (8.2) involves a complex addition, or 2 real additions; and thus a total of about $6N^5$ real additions.

Since values of N greater than 10 occur in practice (for example, in meteorology, Baer & Platzmann (1961) use $N = 19$; Elsaesser (1966*a*) uses $N = 18$), it is important to reduce the preceding estimate. Furthermore, the preceding estimate assumes that the coupling integral in (8.2) has been tabulated beforehand; and although Wigner selection rules and symmetries greatly reduce the storage required, the number of storage locations scales like N^5.

The alternative approach is to generalize Orszag's method. To do this, we first use (3.19) to introduce

$$F(\theta,\phi) = \boldsymbol{a}_{(1)}\,\boldsymbol{b}_{(1)}\cdot\overline{\boldsymbol{e}'_{\mu_1}\boldsymbol{e}'_{\mu_2}},$$

$$= \left[\sum_{m_\alpha=-N}^{N} A_{m_\alpha}(\theta,\mu_2)\,\mathrm{e}^{\mathrm{i}m_\alpha\phi}\right]\left[\sum_{m_\beta=-N}^{N} B_{m_\beta}(\theta,\mu_1)\,\mathrm{e}^{\mathrm{i}m_\beta\phi}\right], \tag{8.4}$$

where
$$A_m(\theta,\mu) = \sum_{n=|m|}^{N}\sum_{n_1}(-)^{n+\mu}\Lambda(n(1))\begin{pmatrix} n & n_1 & 1\\ \mu & 0 & -\mu\end{pmatrix} a_{n(1)}\,d^n_{m,\mu}(\theta), \tag{8.5}$$

and B_m is the same but with $a_{n(1)}$ replaced by $b_{n(1)}$. Secondly, introduce

$$y^m_n(\mu_1,\mu_2) = \sum_{n_1,\,n_2}(-)^{n+n_1+\mu_1}\Lambda(n(2))\begin{pmatrix} n & n_1 & 1\\ \mu_1+\mu_2 & -\mu_2 & -\mu_1\end{pmatrix}\begin{pmatrix} n_1 & n_2 & 1\\ \mu_2 & 0 & -\mu_2\end{pmatrix} x_{n(2)}. \tag{8.6}$$

From (3.19) and (8.1),

$$F(\theta,\phi) = \sum_{m=0}^{2N} g_m(\theta,\mu_1,\mu_2)\,\mathrm{e}^{\mathrm{i}m\phi}, \tag{8.7}$$

where
$$g_m = \sum_{n=|m|}^{2N} y^m_n\,d^n_{m,\mu_1+\mu_2}(\theta). \tag{8.8}$$

Next, following Orszag (1970), introduce a (θ,ϕ)-grid of points

$$\theta_j(j = 1,\ldots,2N), \qquad \phi_l(l = 1,\ldots,4N).$$

To employ the Fast Fourier Transform algorithm (F.F.T.), and to invert (8.7), choose $\phi_l = \pi l/2N$. Now proceed to compute $x_{n(2)}$ as follows:

(1) Assume that the quantities

$$(-)^{n+\mu}\Lambda(n(1))\begin{pmatrix} n & n_1 & 1\\ \mu & 0 & -\mu\end{pmatrix} d^n_{m,\mu}(\theta_j)$$

have been tabulated beforehand, requiring about $9N^3$ real storage locations. Note that negative m need not be considered since

$$A_{-m}(\theta,\mu) = (-)^\mu\overline{A_m}(\theta,-\mu),$$

reflecting (3.4) and a symmetry property of the rotation matrix element $d^n_{m,\mu}(\theta)$. Calculate $A_m(\theta_j,\mu)$ from (8.5) for $\mu = 0, \pm 1$; $|n-1| \leqslant n_1 \leqslant n$; $j = 1,\ldots,2N$; $m = 0,\ldots,N$. This requires at most

$$3^2\times 2N\sum_{m=0}^{N}(N-m+1) \sim 9N^3$$

real $\times$ complex multiplications, each followed by a complex addition. Repeating the calculation for B_m leads to a total of about $36N^3$ real multiplications plus additions.

(2) Use the F.F.T. to compute $F(\theta_j,\phi_l)$ in (8.4). The F.F.Ts of A_m and B_m in (8.4) each require about $4N\log_2 N$ real multiplications plus additions for $l = 1,\ldots,4N$. Letting $j = 1,\ldots,2N$ and $\mu_1,\mu_2 = 0,\pm 1$, results in a total of about $48N^2\log_2 N$ real multiplications plus additions.

(3) Use the identity

$$\frac{1}{4N}\sum_{l=1}^{4N} \mathrm{e}^{\mathrm{i}\pi(m-m')l/2N} = \delta_m^{m'},$$

to invert (8.7) over the (θ, ϕ)-grid, obtaining

$$g_m(\theta_j, \mu_1, \mu_2) = \frac{1}{4N}\sum_{l=1}^{4N} F(\theta_j, \phi_l)\, \mathrm{e}^{-\mathrm{i}m\pi l/2N}.$$

F.F.T. computation of $g_m(\theta_j, \mu_1, \mu_2)$ for $0 \leqslant m \leqslant N$ requires about $4N\log_2 N$ real multiplications plus additions. Letting $j = 1, \ldots, 2N$ and $\mu_1, \mu_2 = 0, \pm 1$ gives a total of about $72N^2\log_2 N$ real multiplications plus additions. Negative m need not be considered since

$$g_{-m}(\theta, \mu_1, \mu_2) = (-)^{\mu_1+\mu_2}\overline{g_m}(\theta, -\mu_1, -\mu_2).$$

(4) Consider (8.8) as representing a set of simultaneous linear equations over the θ-grid. Assume that the relevant inverse matrices, depending only on the quantities $d^n_{m,\mu_1+\mu_2}(\theta_j)$ have been tabulated beforehand, requiring about $12N^3$ real locations. Our only restriction on the choice of θ_j is that these inverse matrices do exist. Use these inverse matrices to compute y^m_n for $\mu_1 = 0, \pm 1$; $\mu_2 = 0, \pm 1$; $0 \leqslant m \leqslant n \leqslant N$. For each N the number of linear equations involved is $2N-m+1$. Thus, using the inverse-matrix coefficients to compute y^m_n for $\mu_1 = 0, \pm 1$; $\mu_2 = 0, \pm 1$; $m = 0, \ldots, N$; $n = m, \ldots, N$; requires

$$3^2 \sum_{m=0}^{N} (2N-m+1)(N-m+1) \sim \tfrac{15}{2}N^3$$

real × complex multiplications, each followed by a complex addition. The total number of real multiplications plus additions is thus about $15N^3$. Again, negative m has not been considered since

$$y^{-m}_n(\mu_1, \mu_2) = (-)^m\overline{y^m_n}(-\mu_1, -\mu_2).$$

(5) Use the orthogonality of 3-j coefficients to invert (8.6), giving

$$x_{n(2)} = \sum_{\mu_1,\mu_2} (-)^{n+n_1+\mu_1}\frac{\Lambda(n_1,n_2)}{\Lambda(n)}\begin{pmatrix} n & n_1 & 1 \\ \mu_1+\mu_2 & -\mu_2 & -\mu_1\end{pmatrix}\begin{pmatrix} n_1 & n_2 & 1 \\ \mu_2 & 0 & -\mu_2\end{pmatrix} y^m_n(\mu_1,\mu_2).$$

Assuming that the coefficients in this sum have been calculated beforehand, requiring about $27N$ real locations, computation of $x_{n(2)}$ for $n = 0, \ldots, N$; $|n-1| \leqslant n_1 \leqslant n+1$; $|n_1-1| \leqslant n_2 \leqslant n_1+1$; $m = 0, \ldots, n$; requires at most

$$3^4 \sum_{n=0}^{N} (n+1) \sim \frac{3^4}{2}N^2$$

real × complex multiplications each followed by a complex addition. That is, about $81N^2$ real multiplications plus additions. Negative m has not been considered because of (3.4).

The grand total of real multiplications plus additions required in this generalization of Orszag's method is about $51N^3 + 120N^2\log_2 N$. For $N \geqslant 3$ this is a large improvement over the $18N^5$ operations previously estimated for computing $x_{n(2)}$ directly from (8.2). Furthermore, permanent storage requirements now also scale like N^3 rather than N^5.

Note finally that the coefficient y^m_n, calculated in step (4) above, is actually the spectral coefficient of $\boldsymbol{a}_{(1)}\boldsymbol{b}_{(1)}$ in the Burridge formalism (see § 6). Thus step (5) above indicates a numerical advantage of the Burridge formalism, amounting to about $40N^2$ real multiplications plus additions. This saving is typically only about 5 % of the grand total of arithmetic operations.

APPENDIX A. POLAR COMPONENTS OF $Y_{n(1)}$ AND $Y_{n(2)}$

Rotation matrix elements and scalar spherical harmonics are related via the formula

$$D^n_{M,\pm m}(-\gamma,-\beta,-\alpha) = e^{i(M\gamma\pm m\alpha)} d^n_{\pm m,M}(\beta)$$
$$= (-)^M\left[\frac{(n-m)!}{(2n+1)!\,(n+m)!}\right]^{\frac{1}{2}} L^m_\pm Y^m_n(\beta,\gamma). \qquad (A\ 1)$$

Here, $L^m_\pm$ represents m successive applications of the differential operators defined by

$$L_\pm \equiv -i\,e^{\pm i\alpha}\left(-\cot\beta\frac{\partial}{\partial\alpha}\pm i\frac{\partial}{\partial\beta}+\operatorname{cosec}\beta\frac{\partial}{\partial\gamma}\right).$$

Introduce the abbreviations

$$Y = Y^m_n(\theta,\phi),\quad E = \partial Y/\partial\theta,\quad F = \partial^2 Y/\partial\theta^2,\quad G = \operatorname{cosec}\theta\,\partial Y/\partial\phi,\quad H = \partial G/\partial\theta.$$

Then, putting $\alpha = 0$ in (A 1) and letting m take the values 0, 1, 2, yields equations equivalent to

$$d^n_{m,0}(\theta)\,e^{im\phi} = Y,$$
$$d^n_{m,-1}(\theta)\,e^{im\phi} = [n(n+1)\,(2n+1)]^{-\frac{1}{2}}\,(E+iG),$$
$$d^n_{m,1}(\theta)\,e^{im\phi} = [n(n+1)\,(2n+1)]^{-\frac{1}{2}}(-E+iG),$$
$$d^n_{m,-2}(\theta)\,e^{im\phi} = \left[\frac{(n-2)!}{(2n+1)\,(n+2)!}\right]^{\frac{1}{2}}[2F+2iH+n(n+1)\,Y],$$
$$d^n_{m,2}(\theta)\,e^{im\phi} = \left[\frac{(n-2)!}{(2n+1)\,(n+2)!}\right]^{\frac{1}{2}}[2F-2iH+n(n+1)\,Y].$$

Substituting these results into (3.19), with $k = 1$ and $k = 2$, gives the polar forms of $Y_{n(1)}$ and $Y_{n(2)}$:

$$[n(2n+1)]^{\frac{1}{2}}\,\boldsymbol{Y}_{n,n-1} = nY\boldsymbol{e}_r+E\boldsymbol{e}_\theta+G\boldsymbol{e}_\phi,$$
$$[n(n+1)]^{\frac{1}{2}}\,\boldsymbol{Y}_{n,n} = iG\boldsymbol{e}_\theta-iE\boldsymbol{e}_\phi,$$
$$[(n+1)\,(2n+1)]^{\frac{1}{2}}\,\boldsymbol{Y}_{n,n+1} = -(n+1)\,Y\boldsymbol{e}_r+E\boldsymbol{e}_\theta+G\boldsymbol{e}_\phi,$$
$$[n(n-1)\,(2n-1)\,(2n+1)]^{\frac{1}{2}}\,\boldsymbol{Y}_{n,n-1,n-2}$$
$$= n(n-1)\,Y\boldsymbol{e}_r\boldsymbol{e}_r+(n-1)\,E\boldsymbol{e}_r\boldsymbol{e}_\theta+(n-1)\,G\boldsymbol{e}_r\boldsymbol{e}_\phi+(n-1)\,E\boldsymbol{e}_\theta\boldsymbol{e}_r$$
$$+(nY+F)\,\boldsymbol{e}_\theta\boldsymbol{e}_\theta+H\boldsymbol{e}_\theta\boldsymbol{e}_\phi+(n-1)\,G\boldsymbol{e}_\phi\boldsymbol{e}_r+H\boldsymbol{e}_\phi\boldsymbol{e}_\theta-(F+n^2Y)\,\boldsymbol{e}_\phi\boldsymbol{e}_\phi,$$
$$in[(n-1)\,(2n+1)]^{\frac{1}{2}}\,\boldsymbol{Y}_{n,n-1,n-1}$$
$$= (1-n)\,G\boldsymbol{e}_\theta\boldsymbol{e}_r-H\boldsymbol{e}_\theta\boldsymbol{e}_\theta+(F+n^2Y)\,\boldsymbol{e}_\theta\boldsymbol{e}_\phi$$
$$+(n-1)\,E\boldsymbol{e}_\phi\boldsymbol{e}_r+(F+nY)\,\boldsymbol{e}_\phi\boldsymbol{e}_\theta+H\boldsymbol{e}_\phi\boldsymbol{e}_\phi,$$
$$n[4n^2-1]^{\frac{1}{2}}\,\boldsymbol{Y}_{n,n-1,n} = -n^2Y\boldsymbol{e}_r\boldsymbol{e}_r-nE\boldsymbol{e}_r\boldsymbol{e}_\theta-nG\boldsymbol{e}_r\boldsymbol{e}_\phi+(n-1)\,E\boldsymbol{e}_\theta\boldsymbol{e}_r$$
$$+(nY+F)\,\boldsymbol{e}_\theta\boldsymbol{e}_\theta+H\boldsymbol{e}_\theta\boldsymbol{e}_\phi+(n-1)\,G\boldsymbol{e}_\phi\boldsymbol{e}_r+H\boldsymbol{e}_\phi\boldsymbol{e}_\theta-(F+n^2Y)\,\boldsymbol{e}_\phi\boldsymbol{e}_\phi,$$
$$in[(2n+1)\,(n+1)]^{\frac{1}{2}}\,\boldsymbol{Y}_{n,n,n-1}$$
$$= -nG\boldsymbol{e}_r\boldsymbol{e}_\theta+nE\boldsymbol{e}_r\boldsymbol{e}_\phi+G\boldsymbol{e}_\theta\boldsymbol{e}_r-H\boldsymbol{e}_\theta\boldsymbol{e}_\theta+F\boldsymbol{e}_\theta\boldsymbol{e}_\phi$$
$$-E\boldsymbol{e}_\phi\boldsymbol{e}_r+[F+n(n+1)\,Y]\,\boldsymbol{e}_\phi\boldsymbol{e}_\theta+H\boldsymbol{e}_\phi\boldsymbol{e}_\phi,$$
$$n(n+1)\,\boldsymbol{Y}_{n,n,n} = -E\boldsymbol{e}_\theta\boldsymbol{e}_r+[F+n(n+1)\,Y]\,\boldsymbol{e}_\theta\boldsymbol{e}_\theta+H\boldsymbol{e}_\theta\boldsymbol{e}_\phi-G\boldsymbol{e}_\phi\boldsymbol{e}_r+H\boldsymbol{e}_\phi\boldsymbol{e}_\theta-F\boldsymbol{e}_\phi\boldsymbol{e}_\phi,$$

$$\mathrm{i}(n+1)\,[n(2n+1)]^{\frac{1}{2}}\,\boldsymbol{Y}_{n,\,n,\,n+1}$$
$$= (n+1)\,G\boldsymbol{e}_r\boldsymbol{e}_\theta-(n+1)\,E\boldsymbol{e}_r\boldsymbol{e}_\phi+G\boldsymbol{e}_\theta\boldsymbol{e}_r-H\boldsymbol{e}_\theta\boldsymbol{e}_\theta+F\boldsymbol{e}_\theta\boldsymbol{e}_\phi$$
$$-E\boldsymbol{e}_\phi\boldsymbol{e}_r+[F+n(n+1)\,Y]\,\boldsymbol{e}_\phi\boldsymbol{e}_\theta+H\boldsymbol{e}_\phi\boldsymbol{e}_\phi,$$

$$(n+1)\,[(2n+3)\,(2n+1)]^{\frac{1}{2}}\,\boldsymbol{Y}_{n,\,n+1,\,n}$$
$$= -(n+1)^2\,Y\boldsymbol{e}_r\boldsymbol{e}_r+(n+1)\,E\boldsymbol{e}_r\boldsymbol{e}_\theta+(n+1)\,G\boldsymbol{e}_r\boldsymbol{e}_\phi-(n+2)\,E\boldsymbol{e}_\theta\boldsymbol{e}_r$$
$$+[F-(n+1)\,Y]\,\boldsymbol{e}_\theta\boldsymbol{e}_\theta+H\boldsymbol{e}_\theta\boldsymbol{e}_\phi-(n+2)\,G\boldsymbol{e}_\phi\boldsymbol{e}_r+H\boldsymbol{e}_\phi\boldsymbol{e}_\theta$$
$$-[F+(n+1)^2\,Y]\,\boldsymbol{e}_\phi\boldsymbol{e}_\phi,$$

$$\mathrm{i}(n+1)\,[(n+2)\,(2n+1)]^{\frac{1}{2}}\,\boldsymbol{Y}_{n,\,n+1,\,n+1}$$
$$= (n+2)\,G\boldsymbol{e}_\theta\boldsymbol{e}_r-H\boldsymbol{e}_\theta\boldsymbol{e}_\theta+[F+(n+1)^2\,Y]\,\boldsymbol{e}_\theta\boldsymbol{e}_\phi$$
$$-(n+2)\,E\boldsymbol{e}_\phi\boldsymbol{e}_r+[F-(n+1)\,Y]\,\boldsymbol{e}_\theta\boldsymbol{e}_\phi+H\boldsymbol{e}_\phi\boldsymbol{e}_\phi,$$

$$[(n+1)\,(n+2)\,(2n+1)\,(2n+3)]^{\frac{1}{2}}\,\boldsymbol{Y}_{n,\,n+1,\,n+2}$$
$$= (n+1)\,(n+2)\,Y\boldsymbol{e}_r\boldsymbol{e}_r-(n+2)\,E\boldsymbol{e}_r\boldsymbol{e}_\theta-(n+2)\,G\boldsymbol{e}_r\boldsymbol{e}_\phi-(n+2)\,E\boldsymbol{e}_\theta\boldsymbol{e}_r$$
$$+[F-(n+1)\,Y]\,\boldsymbol{e}_\theta\boldsymbol{e}_\theta+H\boldsymbol{e}_\theta\boldsymbol{e}_\phi-(n+2)\,G\boldsymbol{e}_\phi\boldsymbol{e}_r+H\boldsymbol{e}_\phi\boldsymbol{e}_\theta$$
$$-[F+(n+1)^2\,Y]\,\boldsymbol{e}_\phi\boldsymbol{e}_\phi.$$

Appendix B. Explicit forms of T, S and A

For the purpose of this appendix, let us adopt the notation $\boldsymbol{T}_{n,\,n_1,\,n_2}$, $\boldsymbol{S}_{n,\,n_1,\,n_2}$, $\boldsymbol{A}_{n,\,n_1,\,n_2}$, for the trace, trace-free symmetric and antisymmetric components of the rank-2 harmonic $\boldsymbol{Y}_{n,\,n_1,\,n_2}$. These components are given by (5.10), (5.11) and (5.12) respectively. More explicit forms are obtained by evaluating the relevant 6-j coefficients:

$$\boldsymbol{T}_{n,\,n-1,\,n-2} = \boldsymbol{0},$$

$$\boldsymbol{S}_{n,\,n-1,\,n-2} = \boldsymbol{Y}_{n,\,n-1,\,n-2},$$

$$\boldsymbol{A}_{n,\,n-1,\,n-2} = \boldsymbol{0},$$

$$\boldsymbol{T}_{n,\,n-1,\,n-1} = \boldsymbol{0},$$

$$\boldsymbol{S}_{n,\,n-1,\,n-1} = (2n)^{-1}\{(n+1)\,\boldsymbol{Y}_{n,\,n-1,\,n-1}+(n^2-1)^{\frac{1}{2}}\,\boldsymbol{Y}_{n,\,n,\,n-1}\},$$

$$\boldsymbol{A}_{n,\,n-1,\,n-1} = (2n)^{-1}\{(n-1)\,\boldsymbol{Y}_{n,\,n-1,\,n-1}-(n^2-1)^{\frac{1}{2}}\,\boldsymbol{Y}_{n,\,n,\,n-1}\},$$

$$\boldsymbol{T}_{n,\,n-1,\,n} = \frac{(2n-1)^{\frac{1}{2}}}{3(2n+1)}\{(2n-1)^{\frac{1}{2}}\,\boldsymbol{Y}_{n,\,n-1,\,n}-(2n+1)^{\frac{1}{2}}\,\boldsymbol{Y}_{n,\,n,\,n}+(2n+3)^{\frac{1}{2}}\,\boldsymbol{Y}_{n,\,n+1,\,n}\},$$

$$\boldsymbol{S}_{n,\,n-1,\,n} = [6n(2n+1)]^{-1}\{(n+1)\,(2n+3)\,\boldsymbol{Y}_{n,\,n-1,\,n}+(2n+3)\,[(2n-1)\,(2n+1)]^{\frac{1}{2}}\,\boldsymbol{Y}_{n,\,n,\,n}$$
$$+n[(2n-1)\,(2n+3)]^{\frac{1}{2}}\,\boldsymbol{Y}_{n,\,n+1,\,n}\},$$

$$\boldsymbol{A}_{n,\,n-1,\,n} = \frac{(2n-1)^{\frac{1}{2}}}{2n(2n+1)}\{(n+1)\,(2n-1)^{\frac{1}{2}}\,\boldsymbol{Y}_{n,\,n-1,\,n}-(2n+1)^{\frac{1}{2}}\,\boldsymbol{Y}_{n,\,n,\,n}-n(2n+3)^{\frac{1}{2}}\,\boldsymbol{Y}_{n,\,n+1,\,n}\},$$

$$\boldsymbol{T}_{n,\,n,\,n-1} = \boldsymbol{0},$$

$$\boldsymbol{S}_{n,\,n,\,n-1} = \left(\frac{n-1}{n+1}\right)^{\frac{1}{2}}\boldsymbol{S}_{n,\,n-1,\,n-1},$$

$$\boldsymbol{A}_{n,\,n,\,n-1} = -\left(\frac{n+1}{n-1}\right)^{\frac{1}{2}}\boldsymbol{A}_{n,\,n-1,\,n-1},$$

$$T_{n,n,n} = -\left(\frac{2n+1}{2n-1}\right)^{\frac{1}{2}} T_{n,n-1\ n},$$

$$S_{n,n,n} = \frac{(4n^2-1)^{\frac{1}{2}}}{n+1} S_{n,n-1,n},$$

$$A_{n,n\ n} = -\frac{1}{n+1}\left(\frac{2n+1}{2n-1}\right)^{\frac{1}{2}} A_{n,n-1,n},$$

$$T_{n,n,n+1} = 0,$$

$$S_{n,n,n+1} = \frac{(n+2)^{\frac{1}{2}}}{2(n+1)}\{(n+2)^{\frac{1}{2}} Y_{n,n,n+1} + n^{\frac{1}{2}} Y_{n,n+1,n+1}\},$$

$$A_{n,n,n+1} = \frac{n^{\frac{1}{2}}}{2(n+1)}\{n^{\frac{1}{2}} Y_{n,n,n+1} - (n+2)^{\frac{1}{2}} Y_{n,n+1,n+1}\},$$

$$T_{n,n+1,n} = \left(\frac{2n+3}{2n-1}\right)^{\frac{1}{2}} T_{n,n-1,n},$$

$$S_{n,n+1,n} = \frac{n}{n+1}\left(\frac{2n-1}{2n+3}\right)^{\frac{1}{2}} S_{n,n-1,n},$$

$$A_{n,n+1,n} = -\frac{n}{n+1}\left(\frac{2n+3}{2n-1}\right)^{\frac{1}{2}} A_{n,n-1,n},$$

$$T_{n,n+1,n+1} = 0,$$

$$S_{n,n+1,n+1} = \left(\frac{n}{n+2}\right)^{\frac{1}{2}} S_{n,n,n+1},$$

$$A_{n,n+1,n+1} = -\left(\frac{n+2}{n}\right)^{\frac{1}{2}} A_{n,n,n+1},$$

$$T_{n,n+1,n+2} = 0,$$

$$S_{n,n+1,n+2} = Y_{n,n+1,n+2},$$

$$A_{n,n+1,n+2} = 0.$$

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